Cosmological Parameters Are Dressed

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In the context of the averaging problem in relativistic cosmology, we provide a key to the interpretation of cosmological parameters by taking into account the actual inhomogeneous geometry of the Universe. We discuss the relation between "bare" cosmological parameters determining the cosmological model and the parameters interpreted by observers with a "Friedmannian bias," which are "dressed" by the smoothed-out geometrical inhomogeneities of the surveyed spatial region.

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A considerable body of researchers in cosmology holds the attitude that, in view of ongoing and near future high-precision experiments in observational cosmology, the (to a large extent established) standard model of cosmology will be finally determined. In particular, this attitude is supported by the possibility of precisely and unambigously determining the three numbers that fix the observer's position in the cosmic triangle and characterize the parameters of a Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology [1]. Refinement of observational data, however, must be paraphrased by a refinement of the theoretical cosmological model. In particular, the widely hold working assumption that the standard model idealizes in a dynamically consistent way the real inhomogeneous Universe should be tested and questioned. For instance, the conjecture that the standard model is equivalent to an averaged inhomogeneous model cannot be held (e.g., [2]). Indeed, in this context, the deviations of the average model from the standard model are condensed into a "backreaction effect." It can be explored to understand the influence of structure inhomogeneities on the evolution of the standard model parameters regionally, but it may not impair the robustness of the conceivably simplest cosmological model on the largest scales.

Complementary to this backreaction effect, we, in this Letter, want to elaborate on a key insight into the *interpretation* of cosmological parameters that, so far, has escaped the attention of researchers in cosmology. It adds a new aspect to the discussion of the effect of inhomogeneites on the standard model parameters but, moreover, it provides an answer to the fundamental problem of interpreting cosmological parameters in an inhomogeneous spacetime geometry.

Let us develop a picture that may guide our thoughts. Imagine a finite amount of material mass distributed inhomogeneously within some spatial domain. We simplify the problem by assuming that the astronomical experiment is carried out in a sufficiently shallow survey region, so that the observed objects, within the approximation standards we want to imply, all lie in a "spatial" section and, referring to the space section itself, we assume that the theoretical model already gives us a suitable split into space and time, i.e., a foliation of spacetime. Suppose now that the observer would be able to quantify the observed objects by their amount of material mass, employing, of course, some theoretical considerations, so that the simplest quantity that the astronomical experiment returns is the total amount of material mass contained within the observed portion of the Universe. This in turn determines, up to the normalization by the "Hubble constant" to which we come later, one of the standard cosmological parameters on the scale of the observed region, the *density parameter*, if the amount of mass is divided by the surveyed volume. It is here, where the "interpretation problem" comes into the fore: the "observer's Universe," due to a lack of better standards, is a constant curvature space section given by the standard model. Calculating the average density with the "Friedmannian volume" is, in this picture, considered as the actual source in Friedmann's equation.

One of the reasons for this commonly held view is that Newtonian cosmology is the familiar framework of structure formation models, and the standard (constant) curvature parameter is merely taken to determine the "background" FLRW model, while structures are described within a Euclidean homogeneous space geometry. The careful reader would object that the actual surveyed volume of the spatial domain is not the volume of a constant curvature FLRW domain, but—taking the curvature fluctuations due to the inhomogeneities into account—is rather the volume of the bumpy geometry of the surveyed region. There is an obvious difference between the bare density parameter (the actual material mass density source), and the parameter obtained with a "Friedmannian bias." We may, say, that the latter is dressed by the geometrical inhomogeneities, which the "interpreter" imagines to be smoothed out, so that the averaged material mass density field is actually considered as an average over a homogeneous geometry. We are going to focus on the relation between "bare" and "dressed" parameters.

Effective cosmological models.—To begin with let us recall a central result in connection with the averaging problem. It has recently been shown that a set of "generalized Friedmann equations," which also incorporate structure inhomogeneities, govern the *effective* cosmological evolution [3]. Effective means that the homogeneous-isotropic variables of the FLRW model are replaced by their Riemannian volume averages on some given spatial domain. In relativistic cosmology the generalized Friedmann equations, restricted here to the simplest matter model "irrotational dust" (more general matter models are discussed in [4]), read

$$6H_{\mathcal{B}_0}^2 - 16\pi G\langle\varrho\rangle_{\mathcal{B}_0} - 2\Lambda + \langle\mathcal{R}\rangle_{\mathcal{B}_0} = -\mathcal{Q}_{\mathcal{B}_0}^K, \qquad (1)$$

$$V_{\mathcal{B}_0}^{-2/3}\frac{d}{dt}(\langle\mathcal{R}\rangle_{\mathcal{B}_0}V_{\mathcal{B}_0}^{2/3}) + V_{\mathcal{B}_0}^{-2}\frac{d}{dt}(\mathcal{Q}_{\mathcal{B}_0}^K V_{\mathcal{B}_0}^2) = 0, \qquad (2)$$

where we have defined, on the averaging domain $\mathcal{B}_0$, the *regional Hubble parameter* as $1/3$ of the spatially averaged rate of expansion θ:

$$3H_{\mathcal{B}_0} := \langle\theta\rangle_{\mathcal{B}_0} = \frac{1}{V_{\mathcal{B}_0}}\int_{\mathcal{B}_0}\theta d\mu_g = \frac{\frac{d}{dt}V_{\mathcal{B}_0}}{V_{\mathcal{B}_0}}. \qquad (3)$$

$V_{\mathcal{B}_0} = \int_{\mathcal{B}_0} d\mu_g$ is the volume of the domain of averaging, $d\mu_g$ is the Riemannian volume element associated with the 3-metric g_{ab} of the hypersurface, $\mathcal{R}$ is the intrinsic scalar curvature, $\langle\varrho\rangle_{\mathcal{B}_0} := M_{\mathcal{B}_0}V_{\mathcal{B}_0}^{-1}$ is the average matter density, where $M_{\mathcal{B}_0} = \text{const}$, and $\frac{d}{dt}$ denotes the time derivative in a comoving frame. (Note that the zero subscript indicates that the averaging domain has the original nonaveraged geometry; we shall later also refer to the corresponding smoothed-out domain.)

The explicit source term $\mathcal{Q}_{\mathcal{B}_0}^K$, the *kinematical backreaction*, appearing in the above equations quantifies the deviations of the average model from the standard FLRW model. It is composed of two positive-definite fluctuation terms (see [3]): first, *shear fluctuations* that tend to mimic the presence of a (kinematical) dark matter component decelerating the expansion and, second, *expansion amplitude fluctuations* that tend to mimic a time-dependent positive cosmological term, an accelerating component ("quintessence").

Equation (2) shows that the averaged scalar curvature is coupled to the "backreaction" dynamically, which is not the case in the corresponding Newtonian set of equations [5]. For $\mathcal{Q}_{\mathcal{B}_0}^K = 0$ the set of generalized Friedmann equations is closed, and we have from Eq. (2) $\langle\mathcal{R}\rangle_{\mathcal{B}_0} \propto V_{\mathcal{B}_0}^{-2/3}$ in agreement with the evolution of the spatially constant curvature in the standard model.

Furthermore, in the general model, we may define *regional cosmological parameters* as the following scale-dependent functionals [3]:

$$\Omega_{\mathcal{B}_0}^M := \frac{8\pi G M_{\mathcal{B}_0}}{3V_{\mathcal{B}_0}H_{\mathcal{B}_0}^2}, \quad \Omega_{\mathcal{B}_0}^\Lambda := \frac{\Lambda}{3H_{\mathcal{B}_0}^2}, \quad \Omega_{\mathcal{B}_0}^R := -\frac{\langle\mathcal{R}\rangle_{\mathcal{B}_0}}{6H_{\mathcal{B}_0}^2},$$
$$\text{and } \Omega_{\mathcal{B}_0}^{Q^K} := -\frac{\mathcal{Q}_{\mathcal{B}_0}^K}{6H_{\mathcal{B}_0}^2}, \qquad (4)$$

$$\text{obeying } \Omega_{\mathcal{B}_0}^M + \Omega_{\mathcal{B}_0}^\Lambda + \Omega_{\mathcal{B}_0}^R + \Omega_{\mathcal{B}_0}^{Q^K} = 1. \qquad (5)$$

In contrast to the standard FLRW cosmological parameters there are four players. In the FLRW case there is by definition no kinematical backreaction, $\mathcal{Q}_{\mathcal{B}_0}^K = 0$. Hence, the "effective cosmology" can be determined by a scale-dependent and regional "cosmic quartet" [6] rather than by a global "cosmic triangle" [1].

It is generally agreed that quantitative investigations of the (kinematical) backreaction effect point towards two results: first, the contribution of $\mathcal{Q}_{\mathcal{B}_0}^K$ to the cosmic quartet is quantitatively small in sufficiently large expanding domains of the Universe (it may contribute significantly on cluster scales and below [7] and may be attributed to cosmic variance on large scales); second, the *dynamical* influence of a nonvanishing backreaction on the other (standard) cosmological parameters can—nevertheless—be large, in other words, the values of the standard parameters found on a given hypersurface at an evolved time are, in general, not related to their initial values according to the FLRW model (for an investigation in Newtonian cosmology see [8]).

Dressing cosmological parameters.—Equation (5) forms the basis of a discussion of cosmological parameters as they determine the theoretical model. They may not be, however, directly accessible to observations. Unlike in Newtonian cosmology, where the corresponding equations have a similar form [5], it is not straightforward to compare the above relativistic average model parameters to observational data. The reason is that the volume averages contain information on the actually present *geometrical inhomogeneities* within the averaging domain. In contrast, the "observer's Universe" is described in terms of a Euclidean or constant curvature model. Consequently, the interpretation of observations within the set of the standard model parameters, if extended by the backreaction parameter or not, neglects the geometrical inhomogeneities that (through the Riemannian volume average) are hidden in the average characteristics of the theoretical cosmology. In other words, an averaging procedure in relativistic cosmology is not complete unless we devise a way to also average the geometrical inhomogeneities. Since geometrical fields are tensorial variables for which possible strategies of averaging form the subject of considerable controversy in the relativistic literature, there is some ambiguity in how such an averaging could be implemented.

We have suggested an answer to this problem in [9], and here we wish to exploit our results for comparing the original averaged model of a surveyed region of the

Universe with the geometrically smoothed-out model which governs the interpretation of the observer's data. This turns out to be rather simple and physically clear, so that we think that explaining the details of the smoothing procedure is not mandatory in this Letter. It suffices to say that the idea of smoothing out the geometrical inhomogeneities was implemented in [9] (see also [10] for a preliminary attempt), by designing a smoothing flow on the basis of the geometrical scaling properties of the matter variables. Moreover, such a smoothing was implemented on a regional and Lagrangian basis, i.e., the metric and the matter variables are smoothed on a geodesic domain in such a way as to preserve its material content. Such requirements characterize in a natural way a Ricci deformation flow for the metric [9]. It is perhaps interesting to note that such a flow is extensively studied in the mathematical literature (see, e.g., [11]), where the Ricci flow plays a basic role in mapping a bumpy 3-geometry into a homogeneous geometry.

Let us highlight some results.

According to [9], the picture discussed in the introduction strictly depends on the ratio between two density profiles defined in the averaging domain: one is naturally associated with the actual matter content of the gravitational sources, whereas the other is the mass density corresponding to the matter content in the given region, but now thought of as averaged over a geometrically smoothed-out domain $\overline{\mathcal{B}}$ with homogeneous geometry:

$$\langle \varrho \rangle_{\mathcal{B}_0} = M_{\mathcal{B}_0}/V_{\mathcal{B}_0}, \qquad \langle \varrho \rangle_{\overline{\mathcal{B}}} = M_{\overline{\mathcal{B}}}/V_{\overline{\mathcal{B}}}. \tag{6}$$

Our assumption of the regional smoothing was such that the total masses are the same, so that we infer from (6) that the average density measured with a "Friedmannian bias" is dressed by a *volume effect* due to the difference between the volume of a smoothed region and the actual volume of the bumpy region:

$$\langle \varrho \rangle_{\mathcal{B}_0} = \langle \varrho \rangle_{\overline{\mathcal{B}}} (V_{\overline{\mathcal{B}}}/V_{\mathcal{B}_0}). \tag{7}$$

A further result that explicitly involves the geometrical smoothing flows is formed by the relation between the constant regional curvature in the smoothed model (e.g., a FLRW domain) and the actual regional average curvature in the theoretical cosmology:

$$\overline{\mathcal{R}}_{\overline{\mathcal{B}}} = \langle \mathcal{R} \rangle_{\mathcal{B}_0} (V_{\overline{\mathcal{B}}}/V_{\mathcal{B}_0})^{-2/3} - \mathcal{Q}^R_{\mathcal{B}_0}, \tag{8}$$

where we have introduced a novel measure for the backreaction of geometrical inhomogeneities capturing the deviations from the standard FLRW space section, the *regional curvature backreaction*: $\mathcal{Q}^R_{\mathcal{B}_0} := \int_0^\infty d\beta \{[V_{\mathcal{B}_\beta}(\beta)]/(V_{\overline{\mathcal{B}}})\}\{\frac{1}{3} \langle [\mathcal{R}(\beta) - \langle \mathcal{R}(\beta) \rangle_{\mathcal{B}_\beta}]^2 \rangle_{\mathcal{B}_\beta} - 2 \langle \tilde{\mathcal{R}}^{ab}(\beta) \tilde{\mathcal{R}}_{ab}(\beta) \rangle_{\mathcal{B}_\beta}\}$, with $\tilde{\mathcal{R}}_{ab} := \mathcal{R}_{ab} - \frac{1}{3} g_{ab} \mathcal{R}$ being the trace-free part of the Ricci tensor $\mathcal{R}_{ab}$ in the hypersurface. $\mathcal{Q}^R_{\mathcal{B}_0}$, built from scalar invariants of the intrinsic curvature, appears to have a similar form as the "kinematical backreaction" term (that was built from invariants of the extrinsic curvature). It features two positive-definite parts, the *scalar curvature amplitude fluctuations* and *fluctuations in metrical anisotropies*. Depending on which part dominates we obtain an under- or overestimate of the actual averaged scalar curvature, respectively. β parametrizes integral curves of the smoothing flow for the metric, so that the expression above indeed refers to the explicit form of this flow. Notwithstanding, this term may be estimated by the actual curvature fluctuations, since the Ricci flow acts in a controllable way such that the maxima of the curvature inhomogeneities are monotonically decreasing during the deformation.

From Eq. (8) we can understand the physical content of geometrical averaging. It makes transparent that, in the smoothed model, the averaged scalar curvature is dressed both by the *volume effect* mentioned above, and by the *curvature backreaction effect* itself. The volume effect is expected precisely in the form occurring in (8), if we think of comparing two regions of distinct volumes, but with the same matter content, in a constant curvature space. Whereas the backreaction term encodes the deviation of the averaged scalar curvature from a constant curvature model, e.g., a FLRW space section.

The bare quartet.—The results discussed above allow us to relate the actual parameters (4) to the values of such parameters obtained as regional averages on a homogeneous geometry by the smoothing procedure. We have seen that a Friedmannian bias in modeling the real observed region of the Universe with a smooth matter distribution evolving in a homogeneous and isotropic geometry, inevitably "dresses" the matter density $\langle \varrho \rangle_{\overline{\mathcal{B}}}$, the Hubble parameter $\overline{H}_{\overline{\mathcal{B}}}$, and the scalar curvature $\overline{\mathcal{R}}_{\overline{\mathcal{B}}}$ with correction factors. Correspondingly, an observer with a Friedmannian bias would interpret his measurements in terms of the dressed cosmological parameters:

$$\overline{\Omega}^M_{\overline{\mathcal{B}}} := \frac{8\pi G M_{\overline{\mathcal{B}}}}{3 V_{\overline{\mathcal{B}}} \overline{H}^2_{\overline{\mathcal{B}}}}, \qquad \overline{\Omega}^\Lambda_{\overline{\mathcal{B}}} := \frac{\Lambda}{3 \overline{H}^2_{\overline{\mathcal{B}}}}, \qquad \overline{\Omega}^R_{\overline{\mathcal{B}}} := -\frac{\overline{\mathcal{R}}_{\overline{\mathcal{B}}}}{6 \overline{H}^2_{\overline{\mathcal{B}}}},$$
$$\text{and} \quad \overline{\Omega}^{Q^K}_{\overline{\mathcal{B}}} := -\frac{\overline{\mathcal{Q}}^K_{\overline{\mathcal{B}}}}{6 \overline{H}^2_{\overline{\mathcal{B}}}}, \tag{9}$$

obeying $\overline{\Omega}^M_{\overline{\mathcal{B}}} + \overline{\Omega}^\Lambda_{\overline{\mathcal{B}}} + \overline{\Omega}^R_{\overline{\mathcal{B}}} + \overline{\Omega}^{Q^K}_{\overline{\mathcal{B}}} = 1.$ (10)

The latter equation follows from our assumption that the smoothing procedure requires one to respect the Hamiltonian constraint of Einstein's equations. Introducing the dimensionless parameters

$$\nu := \frac{V_{\overline{\mathcal{B}}}}{V_{\mathcal{B}_0}}, \qquad \alpha := \frac{\overline{H}^2_{\overline{\mathcal{B}}}}{H^2_{\mathcal{B}_0}}, \qquad \mu := \frac{\mathcal{Q}^R_{\mathcal{B}_0}}{\overline{\mathcal{R}}_{\overline{\mathcal{B}}}}, \tag{11}$$

we can formally study fractions of bare and dressed parameters (making sure that the denominators are nonzero, which is the case in generic situations):

$$\frac{\Omega^M_{\mathcal{B}_0}}{\overline{\Omega}^M_{\overline{\mathcal{B}}}} = \alpha\nu, \qquad \frac{\Omega^\Lambda_{\mathcal{B}_0}}{\overline{\Omega}^\Lambda_{\overline{\mathcal{B}}}} = \alpha, \tag{12}$$

$$\frac{\Omega^R_{\mathcal{B}_0}}{\overline{\Omega}^R_{\overline{\mathcal{B}}}} = \alpha\frac{\langle\mathcal{R}\rangle_{\mathcal{B}_0}}{\overline{\mathcal{R}}_{\overline{\mathcal{B}}}} = \alpha\nu^{2/3}(1+\mu), \qquad \frac{\Omega^{\mathcal{Q}^K}_{\mathcal{B}_0}}{\overline{\Omega}^{\mathcal{Q}^K}_{\overline{\mathcal{B}}}} = \alpha\frac{\mathcal{Q}^K_{\mathcal{B}_0}}{\overline{\mathcal{Q}}^K_{\overline{\mathcal{B}}}}.$$

The above listed relations appear to provide a formal recipe for interpreting the cosmological parameters. Let us illustrate them by considering mixed fractions of various cosmological parameters in order to eliminate, say, the fraction of the Hubble parameters α, and conclude on the values of the others:

$$\frac{\Omega^M_{\mathcal{B}_0}}{\Omega^R_{\mathcal{B}_0}} = \frac{\overline{\Omega}^M_{\overline{\mathcal{B}}}}{\overline{\Omega}^R_{\overline{\mathcal{B}}}}\frac{\nu^{1/3}}{1+\mu}, \qquad \frac{\Omega^M_{\mathcal{B}_0}}{\Omega^\Lambda_{\mathcal{B}_0}} = \frac{\overline{\Omega}^M_{\overline{\mathcal{B}}}}{\overline{\Omega}^\Lambda_{\overline{\mathcal{B}}}}\nu. \tag{13}$$

Reflecting the contemporary view on the cosmological parameters, we may consider a region of the Universe on a sufficiently large scale of the order of 1 Gpc/h. The (possibly also dressed) observations of the first doppler peak in the cosmic microwave background fluctuations at the "Friedmannian scale" $\approx$ 100 Mpc/h favor an approximately vanishing average curvature $\overline{\mathcal{R}}_{\overline{\mathcal{B}}} \approx 0$. Let us, for simplicity, approximate both the bare and the dressed kinematical backreaction parameters by zero. If, again for simplicity, we approximate also the curvature backreaction parameter by zero, $\mu \approx 0$ (in the sense that there are curvature fluctuations present, but the two positive-definite parts compensate each other), we would have an approximately vanishing average curvature also in the actual cosmological model. Then, the standard argument requires compensation of the observed matter content (including dark baryonic and possibly dark nonbaryonic matter components), obeying the commonly agreed upper bound $\overline{\Omega}^M_{\overline{\mathcal{B}}} \lesssim 0.3$ with a cosmological term $\overline{\Omega}^\Lambda_{\overline{\mathcal{B}}} \approx 0.7$. For the bare parameters we then obtain $\Omega^M_{\mathcal{B}_0}/\Omega^\Lambda_{\mathcal{B}_0} \approx \frac{0.3}{0.7}\nu$, which yields the estimate:

$$\Omega^M_{\mathcal{B}_0} \approx \frac{0.3}{0.7}\nu\bigg/\left(1+\frac{0.3}{0.7}\nu\right), \qquad \Omega^\Lambda_{\mathcal{B}_0} \approx 1 - \Omega^M_{\mathcal{B}_0}. \tag{14}$$

This certainly oversimplified example shows that, instead of postulating the presence of a large cosmological term, the bare mass parameter could still acquire values close to 1, if "undressed," and if the volume fraction ν is substantially greater than 1. The second relation in Eq. (12) then shows that the actual Hubble parameter would be larger than the dressed one.

A quantitative estimate that gives us an idea of the order of magnitude of such an effect has been worked out by Hellaby [12] comparing spherically symmetric with FLRW solutions. He employs "volume matching" as proposed by Ellis and Stoeger [13] which should, however, amount to a similar effect as a comparison of the models at equal mass. He finds that the spatial averages of the density profiles as compared with the corresponding (fitted) FLRW parameters yield errors in the range 10–30% for realistically modeled clusters and voids.

It appears that the interpretation of relativistic cosmological parameters is far from trivial, given that we neither touched on the issue of averaging on the observer's light cone in which case the discussed effect interacts with the time evolution of the model (compare the detailed suggestion in [13]), nor did we study the smoothing itself in a dynamical setting. As the present discussion shows, a thorough investigation of volumes of realistic cosmological slices as the "simplest" quantity would considerably enhance our theoretical background to approach observational data. As in other fields such as solid state physics, where the study of *surface roughening* is well developed, cosmology could face the necessity of understanding geometrical structure formation, as it had to face the necessity of understanding structure formation on a homogeneous geometry.

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