Creating Maximally Entangled Atomic States in a Bose-Einstein Condensate

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(Received 15 August 2002; published 23 January 2003)

We propose a protocol to create maximally entangled pairs, triplets, quartiles, and other clusters of Bose-condensed atoms starting from a condensate in the Mott insulator state. The essential element is to drive single atom Raman transitions using laser pulses. Our scheme is simple, efficient, and can be readily applied to the recent experimental system as reported by M. Greiner *et al.* **413**, 44 (2002).

DOI: 10.1103/PhysRevLett.90.030402 PACS numbers: 03.75.Kk, 03.65.Ud, 03.75.Hh, 42.50.–p

The physics of quantum degenerate atomic gases continues in its rapid pace of development, and remains one of the most active research areas in recent years [1]. Increasingly, theoretical and experimental attentions are directed towards the underlying quantum correlation properties of the condensed atoms. It seems likely that such quantum states of matter might prove to be a fertile ground for exploring quantum information science applications.

Recently, a quantum phase transition was observed in a system of Bose-condensed atoms immersed in a periodic array of optical potentials [2]. As expected, when expressed in the simple Bose-Hubbard form [3], the ground state of such a system is controlled by essentially two parameters: (i) the on-site atom-atom interaction u for atoms in the same spatial mode of each individual optical well; and (ii) the nearest neighboring well (single) atom tunneling rate J (taken as positive). When $J \gg |u|$, the condensate ground state is in the usual superfluid (delocalized single atom) state. On the other hand, a Mott insulator state arrives in the opposite limit $|u| \gg J$. In a Mott state, atoms are localized inside individual wells. The condensate ground state takes the form of a product of Fock states with an integer number of atoms on each site. The transition from superfluid to Mott insulator is predicted to occur at $|u|/J \geq \mathsf{z} \times 2.6$ with z the number of nearest neighbors in the periodic well lattice [3,4].

The experimental system that yielded the first clear demonstration of the superfluid/Mott-insulator transition enables individual tuning of the values for both J and u [2]. In the experiment, the average occupations per well were around 1–3 atoms, which could potentially form elementary building blocks for atomic qubit based quantum computing designs [3].

In this paper, we propose to create massive maximum entangled pairs, triplets, quartiles, and other clusters of Bose-condensed atoms in a Mott insulator state. The resulting entanglement, with respect to electronic internal states, is stable and long lived. In the experiment [2], ^{87}Rb atoms in the magnetic trapping state $|a\rangle = |F = 2, M_F = -2\rangle$ were used. Other internal states can be trapped in the same experimental setup. In the simple model to be presented below, a second internal state $|b\rangle$ that can be coupled to $|a\rangle$ through atomic Raman transitions is assumed [2] (as seen in earlier JILA experiments with ^{87}Rb states $|F = 2, M_F = -1\rangle$ and $|F = 1, M_F = 1\rangle$ [5]).

In a Mott state, the system dynamics is rather simple as there exists a fixed (small) number of atoms within each well. If we use the second quantized operators $a(a^\dagger)$ and $b(b^\dagger)$ for atoms in the two internal states, the effective Hamiltonian for each well can be expressed as [6]

$$H = uJ_z^2 + \Omega J_y. \tag{1}$$

The second term denotes the single atom Raman coupling due to external laser fields with a (real) effective Rabi frequency $\Omega(t)$ [7]. The angular momentum operators are the Schwinger representation in terms of the two boson modes:

$$J_x = \tfrac{1}{2}(b^\dagger a + a^\dagger b), \qquad J_y = -\frac{i}{2}(b^\dagger a - a^\dagger b), \qquad J_z = \tfrac{1}{2}(b^\dagger b - a^\dagger a). \tag{2}$$

In the context of SU(2) coherent states of an atomic ensemble, these operators have been used extensively for discussing spin squeezing and other properties of multiatom nonclassical states [8–11]. In particular, as was studied by Molmer and Sorensen [12], an interaction of the type uJ_x^2 generates a maximum entangled N-GHZ (Greenberger-Horne-Zeilinger) state [13] starting from all atoms in state $|a\rangle$ or $|b\rangle$. This has led to the recent creation of a four-ion maximum entangled state [14].

Before we discuss our proposal, we summarize the dynamic generation of a maximum entangled state from the uJ_x^2 interaction. For simplicity, we assume N is even. A maximum entangled N-GHZ state can be written as [12]

$$|\mathrm{GHZ}\rangle_N = \frac{1}{\sqrt{2}}\left(e^{i\phi_b}\frac{b^{\dagger N}}{\sqrt{N!}} + e^{i\phi_a}\frac{a^{\dagger N}}{\sqrt{N!}}\right)|0\rangle$$
$$= \frac{1}{2^{(N+1)/2}\sqrt{N!}}\sum_{m=-(N/2)}^{N/2} C_N^{(N/2)+m} d^{\dagger(N/2)+m} c^{\dagger(N/2)-m}[e^{i\phi_b} + e^{i\phi_a}(-1)^{(N/2)-m}]|0\rangle, \quad (3)$$

where new bosonic operators $d/c = (b \pm a)/\sqrt{2}$ were introduced along with its inverse $b/a = (d \pm c)/\sqrt{2}$. C_N^M is the binomial coefficient. Starting from all atoms in state $|a\rangle$, i.e., with $|\psi(0)\rangle = a^{\dagger N}|0\rangle/\sqrt{N!}$. The state at time t due to a uJ_x^2 interaction alone is

$$|\psi(t)\rangle = \frac{1}{2^{N/2}\sqrt{N!}}\sum_{m=-(N/2)}^{N/2} C_N^{(N/2)+m} d^{\dagger(N/2)+m} c^{\dagger(N/2)-m} e^{-iutm^2}(-1)^{(N/2)-m}|0\rangle, \quad (4)$$

where use has been made of $J_x = (d^\dagger d - c^\dagger c)/2$. To within an overall phase factor $|\psi(\tau)\rangle \equiv |\mathrm{GHZ}\rangle_N$ at $u\tau = (2k+1)\pi/2$ with the shortest time being $\tau = \pi/(2|u|)$. Similarly, starting from state $b^{\dagger N}|0\rangle$ will also arrive at a N-GHZ when $u\tau = (2k+1)\pi/2$ [15].

How could interaction (1) be turned into the required J_x^2 form? Our key observation is that the single atom Raman coupling ΩJ_y generates nothing but a rotation along the y axis. Therefore, we can effectively rotate the J_z^2 term into a J_x^2 term. A similar suggestion was made recently by Jaksch *et al.* [16] in order to tune the overall condensate interaction strength to zero [or SU(2) symmetric].

We therefore suggest operating in a three step protocol in the limit when $|\Omega| \gg N|u|$: (i) Apply a $\pi/2$ pulse $\theta(\tau') = \int_0^{\tau'}\Omega(t)dt = \pi/2$ (of spin 1/2). During this stage the nonlinear interaction can be neglected (because $|\Omega| \gg N|u|$). (ii) Wait for a time $|u|\tau = \pi/2$. (iii) Complete the process by applying a $-\pi/2$ pulse with $\theta(\tau') = -\pi/2$ [e.g., by arranging for $\Omega \to -\Omega$ or by waiting for a $3\pi/2$ pulse as in (i)].

These three steps generate the following effective evolution:

$$U(2\tau' + \tau) \approx e^{i(\pi/2)J_y} e^{-i(\pi/2)J_z^2} e^{-i(\pi/2)J_y} = e^{-i(\pi/2)J_x^2}, \quad (5)$$

i.e., J_z^2 is rotated by $\pi/2$ into J_x^2. From a wide range of numerical simulations, we find that N-GHZ states with extremely high fidelities are realized when $|\Omega|/|u| \geq 50$ for up to four atoms.

While the above scheme works well, it is inherently rather slow. In a two component condensate as assumed, we denote the three relevant scattering lengths as a_{aa}, a_{ab}, and a_{bb}, and assume that motional ground state to be $\psi_{000}(\vec{r}) = \exp[-r^2/(4a_h^2)]/(\sqrt{2\pi}\,a_h)^{3/2}$ of a spherically symmetric harmonic trap $V(\vec{r}) = M\omega_t^2 r^2/2$; we find

$$u = (a_{aa} + a_{bb} - 2a_{ab})\frac{2\pi\hbar^2}{M}\frac{1}{(2\sqrt{\pi}a_h)^3}, \quad (6)$$

with $a_h = \sqrt{\hbar/2M\omega_t}$ the ground state size. For ^{87}Rb, u is very small as $a_{aa} \sim a_{ab} \sim a_{bb}$. When $\omega_t \sim (2\pi)30$ kHz as realized in [2], $|u| \sim (2\pi)20$ Hz if $(a_{aa} + a_{bb} - 2a_{ab})$ is of the order of 1 Å. It takes approximately 10 ms to realize a GHZ state, i.e., in a time significantly shorter than the lifetimes from both the two-body dipolar (> 6 s) and the three-body inelastic collision (> 200 ms) losses with less than five atoms in each well [17].

Another serious experimental concern is that collisions can populate Zeeman states other than $|a\rangle$ or $|b\rangle$. For most systems, this depopulation also occurs on the time scale of $\sim 1/|u|$. It is therefore important to include the full manifold of atomic internal states. To this end, we consider a spinor-1 condensate of ^{87}Rb atoms in its ground state $F = 1$ manifold as realized in the first all optical condensate [18]. If a_{M_F} denotes the bosonic annihilation operator of state $|F = 1, M_F = +, 0, -\rangle$, the ground state Hamiltonian within each well becomes

$$H' = u(L^2 - 2N)$$
$$= u(a_+^\dagger a_+^\dagger a_+ a_+ + a_-^\dagger a_-^\dagger a_- a_- + 2a_+^\dagger a_0^\dagger a_+ a_0 + 2a_-^\dagger a_0^\dagger a_- a_0 - 2a_+^\dagger a_-^\dagger a_+ a_- + 2a_0^\dagger a_0^\dagger a_+ a_- + 2a_+^\dagger a_-^\dagger a_0 a_0), \quad (7)$$

with angular momentum-type operators [19–21]

$$L_+ = \sqrt{2}(a_+^\dagger a_0 + a_0^\dagger a_-), \qquad L_- = L_+^\dagger,$$
$$L_z = a_+^\dagger a_+ - a_-^\dagger a_-, \quad (8)$$

and the number of atoms in the well $N = a_+^\dagger a_+ + a_0^\dagger a_0 + a_-^\dagger a_-$. Although L^2 seems SU(2) symmetric, it is not because L_x, L_y, and L_z are not genuine angular moment operators (for spin-1 atoms); they do not satisfy the Casimir relation $L^2 \neq N(N+1)$ [22]. As was shown previously [19,22], multiatom internal state correlations continue to arise dynamically with H' and the addition of single atom Raman couplings of the type $i\Omega_{\mu\nu}(a_\mu^\dagger a_\nu - a_\nu^\dagger a_\mu)/2$. Unfortunately, we have not been able to solve for the combined dynamics analytically even for a small number of atoms. It is also not apparent how to numerically investigate strategies for creating a N-GHZ state in this case.

Looking back on the two mode model (1) discussed earlier, we realize that, with a constant Ω, a state with two atoms initially in $|b\rangle$ develops into a 2-GHZ state within a time of $\approx \pi/|u|$. Specifically, we find

$$C_{11}(t) = \frac{\Omega}{\sqrt{2}\tilde{\Omega}} e^{i(u/2)t} \sin\tilde{\Omega} t,$$
$$C_{20}(t) = \frac{1}{2} - i\frac{u}{4\tilde{\Omega}} e^{i(u/2)t} \sin\tilde{\Omega} t + \frac{1}{2} e^{i(u/2)t} \cos\tilde{\Omega} t, \qquad (9)$$
$$C_{02}(t) = \frac{1}{2} + i\frac{u}{4\tilde{\Omega}} e^{i(u/2)t} \sin\tilde{\Omega} t - \frac{1}{2} e^{i(u/2)t} \cos\tilde{\Omega} t,$$

with $\tilde{\Omega} = \sqrt{u^2 + 4\Omega^2}/2$ for the coefficients of state vector expansion,

$$|\psi(t)\rangle = C_{20}(t)\frac{1}{\sqrt{2}} b^{\dagger 2}|0,0\rangle + C_{02}(t)\frac{1}{\sqrt{2}} a^{\dagger 2}|0,0\rangle + C_{11}(t) b^\dagger a^\dagger |0,0\rangle. \qquad (10)$$

In the above Eq. (9), we have omitted a common phase factor e^{-iut}. Clearly, $C_{11}(t) = 0$ occurs at

$$2\tilde{\Omega} t_m = \sqrt{1 + 4(\Omega/u)^2}(ut_m) = 2m\pi. \qquad (11)$$

$|\psi(t)\rangle$ becomes a 2-GHZ state (3) when $|C_{20}| = |C_{02}| = 1/\sqrt{2}$. This occurs at $ut_m = (2k+1)\pi$ since $C_{20/02}(t_m) = [1 \pm e^{iut_m/2}(-1)^m]/2$. When $|\Omega/u| \gg 1$, both conditions can be satisfied at different values of t_m and Ω as shown in Fig. 1. The shortest time for a 2-GHZ is then $\sim\pi/|u|$.

Based on this observation, we explored numerically the dynamics of the Hamiltonian $H = u(L^2 - 2N) + i\Omega_{\mu\nu}(a_\mu^\dagger a_\nu - a_\nu^\dagger a_\mu)/2$ assuming a constant $\Omega_{\mu\nu}$ and all atoms initially in the $|+\rangle$ state. As expected, we discovered that maximally entangled states continue to be generated at times $\sim\pi/|u|$ for $N = 2, 3, 4$.

For $N = 2$, we find that we get a 2-GHZ state $(a_+^{\dagger 2} + e^{i\phi} a_{\mu=0,-}^{\dagger 2})|0,0,0\rangle/2$ with either a Raman drive Ω_{+0} or Ω_{+-}. ϕ is a controllable phase shift. The 2-GHZ state occurs at times of $\approx (2k+1)\pi/|u|$ ($\mu = 0$) or $(2k+1)\pi/4|u|$ ($\mu = -$) and also times shifted by a small multiples of $\pi/|\Omega_{+\mu}|$ (when $|\Omega_{+\mu}| \gg |u|$) in their immediate neighborhoods. The state fidelities are always very high as long as k is not too large.

For $N = 3$, only the Ω_{+-} drive seems to create a 3-GHZ state $\propto (a_+^{\dagger 3} + e^{i\phi} a_-^{\dagger 3})|0,0,0\rangle$ at times differing from $\approx (2k+1)\pi/4|u|$ by small multiples of $\pi/|\Omega_{+-}|$. Maximum correlated atomic ensembles in states $|+\rangle$ and $|-\rangle$ were previously predicted to occur due to elastic collisions for an initial condensate in state $|0\rangle$ [23].

For $N = 4$, we find that again only the Ω_{+-} drive allows for a simple identification of a 4-GHZ state $\propto (a_+^{\dagger 4} + e^{i\phi} a_-^{\dagger 4})|0,0,0\rangle$, which also occurs at $\approx (2k+1)\pi/4|u|$ and values shifted by a small multiples of $\pi/|\Omega_{+-}|$ in its neighborhood. Thus, at $t \approx \pi/4|u|$ atoms in wells with $N = 2$ and 4 are both maximum entangled as illustrated in Fig. 2. In this simulation, we have used $\Omega_{+-} = (2\pi)30$ kHz and $u = (2\pi)0.25$ kHz. We note that their respective values are not important except that they scale inversely with the required time. What seems to be important is to assure that $|\Omega_{+-}/u| \geq 100$ for up to four atoms to achieve a high fidelity maximum entangled state.

In conclusion, we have presented a simple and efficient protocol for turning a Mott insulator condensate of ^{87}Rb atoms in the ground state $F = 1$ manifold into a source for maximally entangled atomic clusters. Our protocol is reliable and accessible with current technologies [2]. It produces maximum entangled quantum states of Bose-condensed atoms with high fidelity. The only noticeable drawback seems to be due to the fact that, for ^{87}Rb atoms, $u \propto (a_2 - a_0)$, i.e., the difference of scattering lengths for the two symmetric channels with total spin 0 and 2. Nevertheless, inelastic decay processes are essentially negligible because all spin states of the atomic ground

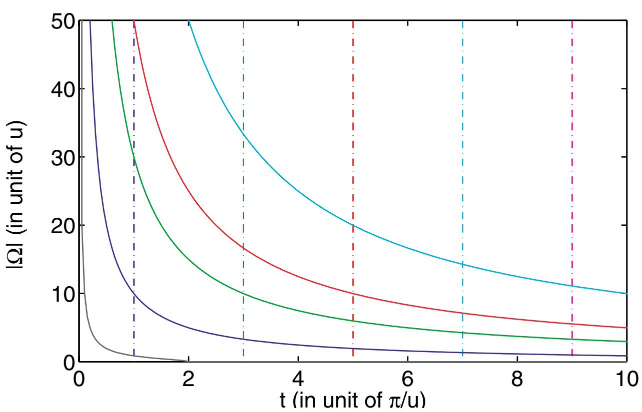

FIG. 1 (color online). Solutions of t_m and Ω as given by the cross points of the two families of curves $(ut_m) = (2k+1)\pi$ (for $k = 0, 1, 2, 3, 4$) and $\sqrt{1 + 4(\Omega/u)^2}(ut_m) = 2m\pi$ (for $m = 1, 10, 30, 50, 100$).

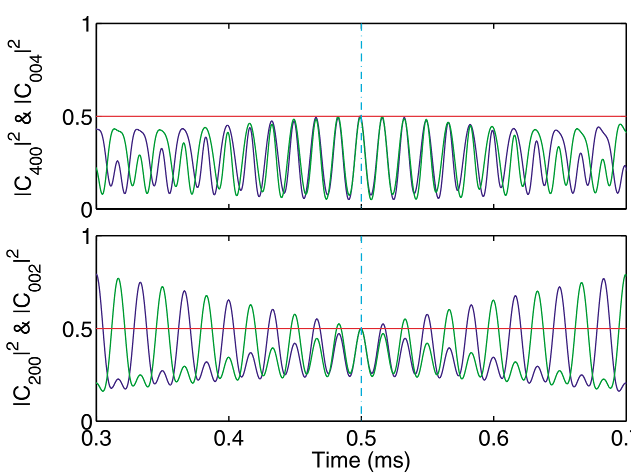

FIG. 2 (color online). The two oscillating lines are, respectively, the probabilities for all atoms in state $|+\rangle$ or $|-\rangle$. Top panel is for $N = 4$, while the bottom one is for $N = 2$ atoms. The vertical dot-dashed line is at $t = 0.5$ (ms).

state manifold are included. Furthermore, the N-GHZ state $(|+\rangle^{\otimes N} + e^{i\phi}|-\rangle^{\otimes N})/\sqrt{2}$ is stable against elastic collisions, which are required to conserve the total M_F, i.e., atoms in the $|+\rangle$ (or $|-\rangle$) state remain in the same state after collisions. Thus, the slow dynamics is perhaps not a major course of concern. Other atomic species (e.g., $F = 1$ manifold of ^{23}Na [24]) may provide large values of u. In Ref. [25], a quantum logic operation between two atoms (one each in two neighboring wells) was proposed that uses the much stronger (by 2 orders of magnitudes) interaction $\propto a_{ab}$. Application of this in a Mott state (with one atom per well) produces GHZ states on a faster time scale, although it requires more complicated internal state dependent optical trapping.

Finally, a condensate in a Mott state contains many individual wells with an identical number of atoms [2]. This makes the experimental detection of the entanglement (for atoms within each well) relatively easy. One can perform the usual parity-type measurement with Ramsey's oscillatory fields technique [14] (again) by driving the single atom Raman transition so quickly that collision effects are negligible. All wells with the same number of atoms thus contribute to the detected signal. Generalizations of our protocol to more than four atoms and other related results will be published elsewhere.

This work is supported by a grant from NSA, ARDA, and DARPA under ARO Contract No. DAAD19-01-1-0667, and by a grant from the NSF PHY-0113831.

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