Quantum Orders in an Exact Soluble Model

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We find all the exact eigenstates and eigenvalues of a spin-1/2 model on square lattice: $H = 16g\sum_i S_i^y S_{i+\hat{x}}^x S_{i+\hat{x}+\hat{y}}^y S_{i+\hat{y}}^x$. We show that the ground states for $g < 0$ and $g > 0$ have different quantum orders described by $Z2A$ and $Z2B$ projective symmetry groups. The phase transition at $g = 0$ represents a new kind of phase transition that changes quantum orders but not symmetry. Both the $Z2A$ and $Z2B$ states contain Z_2 lattice gauge theories at low energies. They have robust topologically degenerate ground states and gapless edge excitations.

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Introduction.—We used to believe that all phases of matter are described by Landau's symmetry breaking theory [1,2]. The symmetry and the related order parameters have dominated our understanding of phases and phase transitions for over 50 years. In this respect, the fractional quantum Hall (FQH) states discovered in 1982 [3,4] opened a new chapter in condensed matter physics. The theory of phases and phase transitions entered into a new era. This is because all different FQH states have the same symmetry and hence cannot be described by Landau's theory. In 1989, it was realized that FQH states, having a robust topological degeneracy, contain a completely new kind of order—topological order [5]. A whole new theory was developed to describe the topological orders in FQH liquids. (For a review, see Ref. [6].)

Landau's theory was developed for classical statistical systems which are described by *positive* probability distribution functions of infinite variables. FQH states are described by their ground state wave functions which are *complex* functions of infinite variables. Thus it is not surprising that FQH states contain addition structures (or a new kind of order) that cannot be described by broken symmetries and Landau's theory. From this point of view, we see that *any* quantum states may contain a new kind of order that is beyond symmetry characterization. Such a kind of order was studied in Ref. [7] and was called quantum order. Since we cannot use order parameter to describe quantum orders, a new mathematical object-projective symmetry group (PSG) was introduced [7] to characterize them. The topological order is a special case of quantum order—a quantum order with a finite energy gap.

One may ask why do we need to introduce a new concept quantum order? What use can it have? To answer such a question, we would like to ask why do we need the concept of symmetry breaking? Is the symmetry breaking description of classical order useful? Symmetry breaking is useful because (a) it leads to a classification of crystal orders, and (b) it determines the structure of low energy excitations without the need to know the details of a system [8,9]. The quantum order and its PSG description are useful in the same sense: (a) PSG allows us to classify over 100 different spin liquids that have the same symmetry [7], and (b) quantum orders determine the structure of low energy excitations without the need to know the details of a system [7,10,11]. The main difference between classical orders and quantum orders is that classical orders produce and protect gapless Nambu-Goldstone modes [8,9] which are a bosonic excitation, while quantum orders can produce and protect gapless gauge bosons and gapless fermions. Fermion excitations (with gauge charge) can even appear in pure bosonic models as long as the boson ground state has a proper quantum order [12–14].

Those amazing properties of quantum orders could fundamentally change our views on the universe and its elementary building blocks. The believed "elementary" particles, such as photons, electrons, etc., may not be elementary after all. Our vacuum may be a bosonic state with a nontrivial quantum order where the elementary gauge bosons and the elementary fermions actually appear as the collective excitations above the quantum ordered ground state. It may be the quantum order that protects the lightness of those elementary particles whose masses are 10^{20} below the natural mass—the Plank mass. Those conjectures are not just wild guesses. A quantum ordered state for a lattice spin model has been constructed [10] which reproduces a complete QED with light, electrons, protons, atoms, etc.

The concept of topological/quantum order is also useful in the field of quantum computation. People have been designing different kinds of quantum entangled states to perform different computing tasks. When a number of qubits becomes larger and larger, it is more and more difficult to understand the pattern of quantum entanglements. One needs a theory to characterize different quantum entanglements in many-qubit systems. The theory of topological/quantum order [6,7] is just such a theory. In fact, topological/quantum orders can be viewed as patterns of quantum entanglements and gauge bosons the fluctuations of quantum entanglements. Also, the robust topological degeneracy in topological ordered states

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discovered in Ref. [5] can be used in fault-tolerant quantum computation [13].

It is hard to convince people about the usefulness of quantum orders when their very existence is in doubt. However, a growing list of soluble or quasisoluble models [13–18] indicates that topological/quantum order does exist beyond FQH states. In particular, Kitaev has constructed exactly soluble spin models that realize both topological orders (i.e., quantum orders with finite energy gap) and gapless quantum orders [14].

In this paper, we study an exact soluble spin-1/2 model on square lattice: $H = 16g \sum_i S_i^y S_{i+\hat{x}x}^x S_{i+\hat{x}+\hat{y}}^y S_{i+\hat{y}}^x$. We find that the ground states for $g < 0$ and $g > 0$ have the same symmetry but different quantum orders. The PSG's for those quantum ordered states are identified. The phase transition at $g = 0$ represents a new kind of phase transition that changes quantum orders but not symmetry. We show that the projective construction that is used to construct quantum ordered ground states [12,19–22] not only gives us exact ground states for our model, but also all the exact excited states. Through this soluble model, we hope to put quantum order and its PSG description on a firm ground.

We would like to mention that the above spin-1/2 model, having one spin per unit cell, is different from Kitaev's exact soluble spin-1/2 models on the links of square lattice and on the sites of honeycomb lattice (which have two spins per unit cell) [13,14]. However, the $g < 0$ version of the above model corresponds to the low energy sector of Kitaev's honeycomb lattice model in the $J_z \gg J_x, J_y$ limit [14,23].

The exact soluble model.—The Hamiltonian of our exact soluble model has a form

$$H = g \sum_i \hat{F}_i, \qquad \hat{F}_i = \tau_i^y \tau_{i+\hat{x}}^x \tau_{i+\hat{x}+\hat{y}}^y \tau_{i+\hat{y}}^x, \tag{1}$$

where $\tau^{x,y,z}$ are the Pauli matrices and $\boldsymbol{i} = (i_x, i_y)$ labels the site of a square lattice. The model is exactly soluble since all the $\hat{F}_i$ operators commute $[\hat{F}_i, \hat{F}_j] = 0$. Since $\hat{F}_i^2 = 1$, the eigenvalues of $\hat{F}_i$ are $F_i = \pm 1$. All the energy eigenstates can be labeled by the sets of common eigenvalues $|\{F_i\}\rangle$. The energy of state $|\{F_i\}\rangle$ is given by $\sum_i gF_i$. We see that when $g < 0$ the ground state has all $F_i = 1$ and when $g > 0$ the ground state has all $F_i = -1$.

The above result is valid only for infinite systems. For finite systems, the situation is much more complicated. On even by even periodic lattice of N_s site, the operators $\hat{F}_i$ satisfy $\prod_{i_x+i_y=\text{even}} \hat{F}_i = 1$ and $\prod_i \hat{F}_i = 1$. Thus, there are only $2^{N_s}/4$ different choices of $\{F_i\}$, which is not enough to label 2^{N_s} different spin states. Later, we will use the slave-boson approach [12,19–22] (or projective construction) to solve the model on finite lattice. We find that the common eigenstates of $\hat{F}_i$ have a fourfold degeneracy. Thus each energy eigenvalue (including the ground state) has at least fourfold degeneracy. The projective construction also relates the fourfold degeneracy of the ground state to an effective Z_2 gauge theory. This allows us to show that the fourfold degeneracy of the ground states is a topological degeneracy which is robust against any local perturbation of the Hamiltonian. The projective construction also allows us to show that the two ground states for $g < 0$ and for $g > 0$ have different quantum orders. It is impossible to change one ground state to the other without phase transitions.

On even by odd periodic lattice, there is only one constraint $\prod_i \hat{F}_i = 1$ and there are $2^{N_s}/2$ different choices of $\{F_i\}$. The projective construction allows us to show that each label $\{F_i\}$ has two degenerate states. Thus the ground states of our model have a twofold topological degeneracy.

Although the $g < 0$ and $g > 0$ ground states share many common properties on an even by even and odd by even lattices, the two states are quite different on an odd by odd lattice. On an odd by odd lattice, the $g < 0$ ground state has an energy $-|g|N_s$ and a twofold degeneracy, while the $g > 0$ ground state, containing a single plaquette with $F_i = 1$ to satisfy the constraint $\prod_i F_i = 1$, has an energy $-|g|(N_s - 2)$ and a $2N_s$-fold degeneracy. The different ground state properties confirm that the $g < 0$ and $g > 0$ ground states have different quantum orders.

Now let us consider a finite $L_x \times L_y$ lattice which is periodic only in the y direction. The lattice has two edges in the y direction. Such a lattice model can be obtained from the periodic lattice model by setting $g = 0$ for a column of plaquettes. We find that the ground states have $\sim 2^{L_y}$-fold degeneracy which correspond to gapless edge states. Since there are $2L_y$ edge sites, we find that there are $\sqrt{2}$ edge states per edge site, indicating that the gapless edge states are described by Majorana fermions.

Exact solution from projective construction.—Usually, the projective construction does not give us exact results. However, our model is constructed such that the projective construction does give us exact results. Our construction is motivated by Kitaev's construction of soluble spin-1/2 models on honeycomb lattice [14]. The key step in both constructions is to find a system of commuting operators. Let $\hat{U}_{ij}^a \equiv \lambda_i^T U_{ij}^a \lambda_j$, where $\boldsymbol{i}, \boldsymbol{j}$ label lattice sites, a is an integer index, U_{ij}^a is an $n \times n$ matrix satisfying $(U_{ij}^a)^T = -U_{ji}^a$, and $\lambda_i^T = (\lambda_{1,i}, \lambda_{2,i}, \ldots, \lambda_{n,i})$ is a n-component Majorana fermion operator satisfying $\{\lambda_{a,i}, \lambda_{b,j}\} = 2\delta_{ab}\delta_{ij}$. We require that all $\hat{U}_{ij}^a$ commute with each other: $[\hat{U}_{i_1 i_2}^a, \hat{U}_{j_1 j_2}^b] = 0$, which can be satisfied if and only if

$$\begin{aligned} &U_{i_1 i_2}^a U_{i_2 i_3}^b = 0, \qquad U_{i_1 i_2}^a U_{i_2 i_1}^b = (U_{i_1 i_2}^a U_{i_2 i_1}^b)^T, \\ &U_{i_1 i_1}^a U_{i_1 i_2}^b = 0, \end{aligned} \tag{2}$$

where $\boldsymbol{i}_1$, $\boldsymbol{i}_2$, and $\boldsymbol{i}_3$ are all different. From a solution of Eq. (2), we find that

$$\hat{U}_{i,i+\hat{x}} = -i\lambda_{1,i}\lambda_{3,i+\hat{x}}, \qquad \hat{U}_{i,i+\hat{y}} = -i\lambda_{2,i}\lambda_{4,i+\hat{y}} \tag{3}$$

form a commuting set of operators.

After obtaining a commuting set of operators, we can easily see that the following Hamiltonian

$$H = g\sum_i \hat{F}_i, \qquad \hat{F}_i = \hat{U}_{i,i_1}\hat{U}_{i_1,i_2}\hat{U}_{i_2,i_3}\hat{U}_{i_3,i} \tag{4}$$

commutes with all the $\hat{U}_{ij}$'s, where $\boldsymbol{i}_1 = \boldsymbol{i} + \hat{\boldsymbol{x}}$, $\boldsymbol{i}_2 = \boldsymbol{i} + \hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}$, and $\boldsymbol{i}_3 = i + \hat{\boldsymbol{y}}$. We will call $\hat{F}_i$ a Z_2 flux operator. Let $|s_{ij}\rangle$ be the common eigenstate of $\hat{U}_{ij}$ with eigenvalue s_{ij}. Since $(\hat{U}_{ij})^2 = 1$, s_{ij} satisfies $s_{ij} = \pm 1$ and $s_{ij} = s_{ji}$. $|s_{ij}\rangle$ is also an energy eigenstate of Eq. (4) with energy

$$E = g\sum_i F_i, \qquad F_i = s_{i,i_1}s_{i_1,i_2}s_{i_2,i_2}s_{i_3,i}. \tag{5}$$

We note that $|s_{ij}\rangle$ is the ground state of the following free fermion system

$$H_{\text{mean}} = -\sum_{\langle ij\rangle}(s_{ij}\hat{U}_{ij} + \text{H.c.}). \tag{6}$$

Let us discuss the Hilbert space within which the Hamiltonian H in Eq. (4) acts. On each site, we group $\lambda_{1,2,3,4}$ into two fermion operators

$$2\psi_{1,i} = \lambda_{1,i} + i\lambda_{3,i}, \qquad 2\psi_{2,i} = \lambda_{2,i} + i\lambda_{4,i}. \tag{7}$$

$\psi_{1,2}$ generates a four dimensional Hilbert space on each site. Let us assume the 2D square lattice to have N_s lattice sites and a periodic boundary condition in both directions. Since there are a total of 2^{2N_s} different choices of s_{ij} (two choices for each link), the states $|s_{ij}\rangle$ exhaust all the 4^{N_s} states in the Hilbert space. Thus the common eigenstates of $\hat{U}_{ij}$ is not degenerate and the above approach allows us to obtain all the eigenstates and eigenvalues of the H.

We note that the Hamiltonian H can only change the fermion number on each site by an even number. Thus the H acts within a subspace which has an even number of fermions on each site. The subspace has only two states per site. When defined on the subspace, H actually describes a spin-1/2 or a hard-core boson system. In fact, within the subspace the fermion Hamiltonian equation (4) becomes our spin-1/2 model Eq. (1).

The subspace is formed by states that are invariant under local Z_2 gauge transformations: $\psi_{Ii} \to G_i\psi_{Ii}$, $G_i = \pm 1$. We will call those states physical states and call the subspace the physical Hilbert space. All the physical states can be obtained from the $|s_{ij}\rangle$ states by projecting into the subspace with even numbers of fermions per site. Since the Z_2 gauge transformations change s_{ij} to $\tilde{s}_{ij} = G_i s_{ij} G_j$, we find $|s_{ij}\rangle$ and $|\tilde{s}_{ij}\rangle$ give rise to the same physical state after projection (if their projection is not zero). We also note that the product of all links $\prod_i s_{i,i+x}s_{i,i+y} = (-)^{\hat{N}_f}$, where $\hat{N}_f = \sum_i(\psi_{1i}^\dagger\psi_{1i} + \psi_{2i}^\dagger\psi_{2i})$ is the total fermion number operator. Thus the projection of $|s_{ij}\rangle$ is nonzero only when $\prod_i s_{i,i+x}s_{i.i+y} = 1$.

The above results allow us to count the number of physical states. Again we assume a periodic boundary condition in both directions. Noting that the constant Z_2 gauge transformation $G_i = -1$ does not change s_{ij}, thus there are $2^{N_s}/2$ distinct s_{ij} that are gauge equivalent to each other. Among 4^{N_s} $|s_{ij}\rangle$ state, $4^{N_s}/2$ of them satisfy $\prod_i s_{i,i+x}s_{i,i+y} = 1$. Thus there are $[(4^{N_s}/2)/(2^{N_s}/2)] = 2^{N_s}$ physical states, which agree with the number of spin-1/2 states in our model. Thus we can obtain all the eigenstates and eigenvalues of the H in the physical Hilbert space from our construction.

Let us assume that one of the eigenstates of our spin model Eq. (1) is given by the projection of a $|s_{ij}\rangle$ state. Other degenerate eigenstate states can be obtained by performing the following two transformations:

$$\begin{aligned} &T_1{:}(s_{i,i+x}, s_{i,i+y}) \to [s_{i,i+x}, (-)^{\delta_{i_y}}s_{i,i+y}],\\ &T_2{:}(s_{i,i+x}, s_{i,i+y}) \to [(-)^{\delta_{i_x}}s_{i,i+x}, s_{i,i+y}]. \end{aligned} \tag{8}$$

We see that T_1 does not change $s_{i,i+x}$ and only flips the sign of $s_{i,i+y}$ when $i_y = 0$. If we view the periodic lattice as a torus, T_1 and T_2 insert π flux through the two holes of the torus. On an even by even lattice, the transformations T_1 and T_2 do not change the product $\prod_i s_{i,i+x}s_{i,i+y}$. Therefore, the three transformations T_1, T_2, and T_1T_2 generate the other three degenerate states. Our spin-1/2 model has four degenerate ground states on an even by even periodic lattice.

On an even by odd lattice, the state generated by T_2 has odd numbers of fermions and does not correspond to any physical spin-1/2 state. Thus, we can only use T_1 to generate the other degenerate state. There are only two degenerate ground states on an even by odd periodic lattice (generated by T_1). On an odd by odd lattice and if $g < 0$, there are also two degenerate ground states generated by T_1T_2.

We note that, locally, the T_1 and T_2 transformations are indistinguishable from Z_2 gauge transformation. Since the physical spin operators are invariant under Z_2 gauge transformation, they are also invariant under T_1 and T_2 transformations. Therefore, the degenerate ground states generated by T_1 and T_2 remain to be degenerate even after we add an arbitrary local perturbation to our exact soluble model Eq. (1). The degeneracy of ground states is a robust topological property, indicating nontrivial topological order in the ground state [5,12].

Different quantum orders in the $g < 0$ and $g > 0$ ground states.—To understand the different quantum/topological orders in the $g < 0$ and $g > 0$ ground states, we need to use the PSG description of quantum order [7]. Here, we will give a brief review of PSG characterization of quantum orders.

Our spin-1/2 model can also be viewed as a hard-core boson model, if we identify $|\downarrow\rangle$ state as a zero-boson state $|0\rangle$ and $|\uparrow\rangle$ state as a one-boson state $|1\rangle$. In the following we will use the boson picture to describe our model.

To construct quantum ordered (or entangled) many-boson wave functions, we will use projective construction. We first introduce a "mean-field" fermion Hamiltonian [7]:

$$H_{\rm mean} = \sum_{\langle ij\rangle} (\psi^\dagger_{I,i}\chi^{IJ}_{ij}\psi_{J,j} + \psi^\dagger_{I,i}\eta^{IJ}_{ij}\psi^\dagger_{J,j} + {\rm H.c.}), \quad (9)$$

where $I, J = 1, 2$. We will use χ_{ij} and η_{ij} to denote the 2×2 complex matrices whose elements are χ^{IJ}_{ij} and η^{IJ}_{ij}. Let $|\Psi^{(\chi_{i,j},\eta_{ij})}_{\rm mean}\rangle$ be the ground state of the above free fermion Hamiltonian, then a many-body boson wave function can be obtained

$$\Phi^{(\chi_{ij},\eta_{ij})}(\boldsymbol{i}_1, \boldsymbol{i}_2 \cdots) = \langle 0| \prod_n b(\boldsymbol{i}_n)|\Psi^{(\chi_{ij},\eta_{ij})}_{\rm mean}\rangle, \quad (10)$$

where

$$b(\boldsymbol{i}) = \psi_{1,i}\psi_{2,i}. \quad (11)$$

According to Ref. [7], the quantum order in the boson wave function $\Phi^{(\chi_{ij},\eta_{ij})}(\{\boldsymbol{i}_n\})$ can be (partially) characterized by PSG. To define PSG, we first discuss two types of transformations. The first type is SU(2) gauge transformation

$$(\psi_i, \chi_{ij}, \eta_{ij}) \to (G_i\psi_i, G_i\chi_{ij}G^\dagger_j, G_i\eta_{ij}G^T_j), \quad (12)$$

where $G_i \in$ SU(2). We note that the physical boson wave function $\Phi^{(\chi_{ij},\eta_{ij})}(\{\boldsymbol{i}_n\})$ is invariant under the above SU(2) gauge transformations. The second type is the usual symmetry transformation, such as the translations T_x: $\boldsymbol{i} \to \boldsymbol{i} - \hat{\boldsymbol{x}}$, T_y: $\boldsymbol{i} \to \boldsymbol{i} - \hat{\boldsymbol{y}}$. A generic transformation is a combination of the above two types, say $GT_x(\chi_{ij}) = G_i\chi_{i-\hat{x},j-\hat{x}}G^\dagger_j$. The PSG for an ansatz (χ_{ij}, η_{ij}) is formed by all the transformations that leave the ansatz invariant.

Every PSG contains a special subgroup, which is called the invariant gauge group (IGG). An IGG is formed by pure gauge transformations that leave the ansatz unchanged $\text{IGG} \equiv \{G|\chi_{ij} = G_i\chi_{ij}G^\dagger_j, \eta_{ij} = G_i\eta_{ij}G^T_j\}$. One can show that PSG, IGG, and the symmetry group (SG) of the many-boson wave function are related: PSG/IGG = SG [7].

Different quantum orders in the ground states of our boson system are characterized by different PSG's. In the following we will concentrate on the simplest kind of quantum orders whose PSG has a IGG $= Z_2$. We will call those quantum states Z_2 quantum states. We would like to ask how many different Z_2 quantum states are there that have translation symmetry. According to our PSG characterization of quantum orders, the above physical question becomes the following mathematical question: how many different PSG's are there that satisfy PSG/Z_2 = translation symmetry group. This problem has been solved in Ref. [7]. The answer is 2 for 2D square lattice. Both PSG's are generated by three elements $\{G^xT_x, G^yT_y, G^g\}$, where G^g is a pure gauge transformation that generates the Z_2 IGG: IGG $= \{1, G^g\}$. The gauge transformations in the three generators for the first Z_2 PSG are given by

$$G^g_i = -1, \qquad G^x_i = 1, \qquad G^y_i = 1. \quad (13)$$

Such a PSG will be called a *Z2A* PSG. The quantum states characterized by *Z2A* PSG will be called *Z2A* quantum states. For the second Z_2 PSG, we have

$$G^g_i = -1, \qquad G^x_i = 1, \qquad G^y_i = (-1)^{i_x}. \quad (14)$$

Such a PSG will be called a *Z2B* PSG.

When $g < 0$, the ground state of our model is given by Z_2 flux configuration $F_i = 1$. To produce such a flux, we can choose $s_{i,i+\hat{x}} = s_{i,i+\hat{y}} = 1$. In this case, Eq. (6) becomes Eq. (9) with $-\eta_{i,i+\hat{x}} = \chi_{i,i+\hat{x}} = 1 + \tau^z$ and $-\eta_{i,i+\hat{y}} = \chi_{i,i+\hat{y}} = 1 - \tau^z$. The PSG for the above ansatz turns out to be the *Z2A* PSG in Eq. (13). Thus the ground state for $g < 0$ is a *Z2A* state. Since IGG $= Z_2$, the low energy effective theory is a Z_2 gauge theory [7].

When $g > 0$, the ground state is given by configuration $F_i = -1$ which can be produced by $(-)^{i_y}s_{i,i+\hat{x}} = s_{i,i+\hat{y}} = 1$. The ansatz now has a form $-\eta_{i,i+\hat{x}} = \chi_{i,i+\hat{x}} = (-)^{i_y}(1 + \tau^z)$ and $-\eta_{i,i+\hat{y}} = \chi_{i,i+\hat{y}} = 1 - \tau^z$. Its PSG is the *Z2B* PSG in Eq. (14). Thus the ground state for $g > 0$ is a *Z2B* state. The $g < 0$ and $g > 0$ ground states have different quantum orders.

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