Monte Carlo Evaluation of Non-Abelian Statistics

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(Received 27 September 2002; published 10 January 2003)

We develop a general framework to (numerically) study adiabatic braiding of quasiholes in fractional quantum Hall systems. Specifically, we investigate the Moore-Read (MR) state at $\nu = 1/2$ filling factor, a known candidate for non-Abelian statistics, which appears to actually occur in nature. The non-Abelian statistics of MR quasiholes is demonstrated explicitly for the first time, confirming the results predicted by conformal field theories.

DOI: 10.1103/PhysRevLett.90.016802 PACS numbers: 73.43.–f, 05.30.–d

The quantum statistics of a system of identical particles describe the effect of adiabatic particle interchange on the many-body wave function. All fundamental particles belong to one of two classes: those that have their wave function unaffected by particle interchange (bosons) and those whose wave function gets a minus sign under permutation (fermions). In two dimensions, it is known that a number of exotic types of statistics can exist for particlelike collective excitations. For example, elementary excitations of the Laughlin fractional quantum Hall (FQH) states exhibit "fractional" statistics: The phase of the wave function is rotated by an odd fraction of π when two Laughlin quasiparticles (or quasiholes) are interchanged [1,2]. Even more exotic statistics can exist when a system with several excitations fixed at given positions is degenerate [3]. In such a case, adiabatic interchange (braiding) of excitations can nontrivially rotate the wave function within the degenerate space. In general, these braiding operations need not commute; hence the statistics are termed "non-Abelian." Remarkably, the Moore-Read (MR) state, a state which is commonly believed [4] to describe observed FQH plateaus at $\nu = 5/2$ and $7/2$ (which correspond, respectively, to half filling of electrons or holes in the first excited Landau level), is thought to have such non-Abelian elementary excitations [3]. Other possible physical realizations of non-Abelian statistics have also been proposed [5]. States of this type have been suggested to be attractive for quantum computation [6].

In Ref. [2], in order to establish the nature of the statistics of the Laughlin quasiholes, a Berry phase calculation was performed that explicitly kept track of the wave-function phase as one quasihole was transported around the other. Although approximations were involved in this calculation, it nonetheless established quite convincingly the fractional nature of the statistics. Unfortunately, it has not been possible to generalize this calculation to explicitly investigate statistics of the MR quasiholes [3]. Although there has been much study of the statistics of the MR quasiholes in the framework of conformal field theories (CFT), it would be desirable to perform a direct calculation analogous to that of Ref. [2]. The purpose of this Letter is to provide such a calculation, albeit numerically. Furthermore, the approach developed here is readily applicable to other FQH systems which are not easily accessible to analytic investigations.

The evolution operator of a many-body system described by a Hamiltonian $H(\lambda)$ is in principle determined by the Schrödinger equation. In general, $H(\lambda)$ itself can change in time through dependence on some varying parameter $\lambda(t)$. In such a case, let us define $\varphi_i(t)$ at a given time t to be an orthonormal basis for a particular degenerate subspace, requiring that this basis is locally smooth as a function of t. If λ is varied adiabatically (and so long as the subspace does not cross any other states), then the time-evolution operator maps an orthonormal basis of the subspace at one t onto an orthonormal basis at another t. A solution of the Schrödinger equation, $\psi_i(t) = U_{ij}(t)\varphi_j(t)$, is simply given by [7]

$$(U^{-1}\dot{U})_{ij} = \langle\varphi_i|\dot{\varphi}_j\rangle \equiv A_{ij}(t). \qquad (1)$$

Since the matrix A is anti-Hermitian, $U(t)$ is guaranteed to be unitary if its initial value $U(0)$ is unitary. Note that if we vary λ so that the Hamiltonian returns to its initial value at time t, i.e., $H[\lambda(t)] = H[\lambda(0)]$, the corresponding transformation of the degenerate subspace can be nontrivial, i.e., $\psi_i(t) \neq \psi_i(0)$ [7].

We explicitly demonstrate that this is the case for the MR state with at least four quasiholes. The analysis is done in spherical geometry [8]: N electrons are positioned on a sphere of unit radius, with their coordinates given by $(u_1, v_1), \ldots, (u_N, v_N)$, using the spinor notation (i.e., $u = e^{i\phi/2}\cos\theta/2$ and $v = e^{-i\phi/2}\sin\theta/2$ in terms of the usual spherical coordinates). A monopole of charge $2S = 2N + n - 3$ in units of the flux quanta $\Phi_0 = hc/e$ is placed in the center of the sphere, giving rise to $2n$ quasiholes which are put at $(\tilde{u}_1, \tilde{v}_1), \ldots, (\tilde{u}_{2n}, \tilde{v}_{2n})$. Using the gauge $\vec{A} = (\Phi_0 S/2\pi)\hat{\phi}\cot\theta$, the MR wave function [3] is then given by

$$\psi_{\rm Pf} = {\rm Pf}\Lambda_{ij}^{(a,b,\ldots)(\alpha,\beta,\ldots)} \prod_{i<j}(u_i v_j - v_i u_j)^2, \qquad (2)$$

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where $\mathrm{Pf}\Lambda_{ij}^{(a,b,\ldots)(\alpha,\beta,\ldots)}$ is the Pfaffian [3] of the $N \times N$ antisymmetric matrix [9]

$$\Lambda_{ij}^{(a,b,\ldots)(\alpha,\beta,\ldots)} = (u_i v_j - v_i u_j)^{-1} \times [(u_i \tilde{v}_a - v_i \tilde{u}_a)(u_j \tilde{v}_\alpha - v_j \tilde{u}_\alpha) \times (u_i \tilde{v}_b - v_i \tilde{u}_b)(u_j \tilde{v}_\beta - v_j \tilde{u}_\beta) \times \cdots + (i \leftrightarrow j)].$$

Pfaffian wave functions (2) were first constructed in Ref. [3] as CFT conformal blocks. This MR state is the *exact* ground state for a special three-body Hamiltonian [11] and is also thought to pertain to realistic two-body interactions in the first excited Landau level [4]. The presence of quasiholes in the ground state is dictated by the incommensuration of the flux with the electron number. Physically, the MR state can be thought of as p-wave BCS pairing of composite fermions at zero net field with quasiholes being the vortex excitations [3,12,13]. Each quasihole has charge $e/4$ and corresponds to half a quantum of flux (because of the paired order parameter [3]). Equation (2) describes a state with quasiholes created in two equal-size groups: $(\tilde{u}_a, \tilde{v}_a)$, $(\tilde{u}_b, \tilde{v}_b)$, ... and $(\tilde{u}_\alpha, \tilde{v}_\alpha)$, $(\tilde{u}_\beta, \tilde{v}_\beta)$, ... Different quasihole groupings realize a space with degeneracy 2^{n-1} [10,14]. (Even though there are $2n!/2(n!)^2$ ways to arrange $2n$ quasiholes into two groups of n, the resulting wave functions are not all linearly independent.) In the presence of finite-range interactions, the exact degeneracy may be split by an amount exponentially small in the large vortex separation [12]. In this case, infinitely slow braiding will not exhibit non-Abelian statistics, although for a very wide range of intermediate time scales, such statistics should apply [12]. The effects of disorder on the statistics are only partially understood [12].

Consider an orthonormal basis φ_i, with $i = 1, \ldots, 2^{n-1}$, for the subspace with $2n$ quasiholes, which is locally smooth when parametrized by the quasihole coordinates. In order to determine the braiding statistics, we find the transformation $\varphi_i \to U_{ij}\varphi_j$ under the evolution operator after two of the quasiholes are interchanged while the others are held fixed. The unitary matrix U_{ij} is obtained by first solving Eq. (1) and then projecting the final basis onto the initial one. (Since we require φ_i to be only locally smooth, the basis itself can nontrivially rotate after the quasiholes return to their original positions.) Equation (1) is integrated numerically: The differential equation is discretized and the wave-function overlaps (the right-hand side of the equation) are evaluated using the Metropolis Monte Carlo method. The computational errors are easily evaluated by varying the number of operations. We aim the calculation at addressing the following questions: (i) What is the Berry phase accumulated upon quasihole interchange due to the enclosed magnetic flux and due to the relative statistics? (ii) What is the transformation matrix for the ground-state subspace corresponding to the braiding operations? In the following, we first describe the numerical method, then present the results, and then compare them to CFT predictions [3,10].

In order to integrate Eq. (1) numerically, the quasihole interchange is performed in a finite number of steps. If $U^{(l)}$ is the value of the transformation matrix at the lth step, then at the next step

$$U^{(l+1)} = U^{(l)}[1 + A^{(l)}/2][1 - A^{(l)}/2]^{-1}, \tag{3}$$

where $A_{ij}^{(l)} = \langle \varphi_i^{(l+1)} + \varphi_i^{(l)} | \varphi_j^{(l+1)} - \varphi_j^{(l)} \rangle / 2$. Our choice of the finite-element scheme (3) will become clear later. In practice, in general we do not know an orthonormal basis for the MR states (2) in an analytic form, but we can numerically orthonormalize a set of 2^{n-1} linearly independent Pfaffian wave functions $\psi_{\mathrm{Pf}i}$. Let $B_{ij}^{(l)} = [\psi_{\mathrm{Pf}i}^{(l)}, \psi_{\mathrm{Pf}j}^{(l)}]$ denote the normalized overlaps of different states. (It is implied here and throughout the Letter that $[\psi_{\mathrm{Pf}i}^{(k)}, \psi_{\mathrm{Pf}j}^{(l)}] \equiv \langle \psi_{\mathrm{Pf}i}^{(k)} | \psi_{\mathrm{Pf}j}^{(l)} \rangle / \| \psi_{\mathrm{Pf}i}^{(k)} \| \| \psi_{\mathrm{Pf}j}^{(l)} \|$ is evaluated numerically.) We then easily show that

$$A^{(l)} = [V^{(l)}]^\dagger W^{(l)} V^{(l+1)}/2 - \mathrm{H.c.}, \tag{4}$$

where $W_{ij}^{(l)} = [\psi_{\mathrm{Pf}i}^{(l)}, \psi_{\mathrm{Pf}j}^{(l+1)}]$ and $V^{(l)}$ is defined by $[V^{(l)}]^\dagger B^{(l)} V^{(l)} = \hat{1}$, constructing an orthonormal basis $\varphi_i^{(l)} = V_{ji}^{(l)} \psi_{\mathrm{Pf}j}^{(l)}$. We require $V^{(l)}$ to be locally smooth as a function of the quasihole coordinates: The basis can continuously transform while the quasiholes are moved, but, e.g., sudden sign flips are not allowed.

According to Eq. (4), $A^{(l)}$ is anti-Hermitian, so that the transformation $U^{(l+1)}$ is guaranteed to be unitary if $U^{(l)}$ is unitary. This explains our choice (3) for discretizing Eq. (1). Another feature preserved by our numerical scheme is that making a step forward, $\psi_{\mathrm{Pf}i}^{(l)} \to \psi_{\mathrm{Pf}i}^{(l+1)}$, followed by a step backward, $\psi_{\mathrm{Pf}i}^{(l+1)} \to \psi_{\mathrm{Pf}i}^{(l)}$, results in a trivial transformation. We start at $U^{(0)} = \hat{1}$ and find $U^{(n_s)}$ after performing $n_s + 1$ steps for braiding of two quasiholes (n_s is increased to convergence). Because $\psi_{\mathrm{Pf}i}^{(n_s)}$ is some nontrivial linear combination of $\psi_{\mathrm{Pf}i}^{(0)}$, we, finally, have to project the transformation onto the initial basis: $U^{(n_s)} \to U^{(n_s)} O^T$, where $O = [V^{(0)}]^\dagger \Omega V^{(n_s)}$ and $\Omega_{ij} = [\psi_{\mathrm{Pf}i}^{(0)}, \psi_{\mathrm{Pf}j}^{(n_s)}]$. The resulting unitary transformation matrix U then gives a representation of the braid group for quasihole interchanges. In the following, we describe our numerical experiments.

The space describing $2n = 2$ MR quasiholes is nondegenerate, so non-Abelian statistics cannot occur. There is, nevertheless, a Berry phase accumulated from wrapping these quasiholes around each other. Our calculation of this phase for the MR state is analogous to the one performed in Ref. [2] for the Laughlin state, except that our calculation is numerical and therefore requires no mean-field approximation. Let us first briefly recall results for the Laughlin wave function at filling factor $\nu = 1/p$. In the disk geometry, the Berry phase χ corresponding to taking a single quasihole around a loop is given by 2π for each enclosed electron, i.e., $\chi = 2\pi\langle N \rangle$, where $\langle N \rangle$ is the expectation number of enclosed electrons [2].

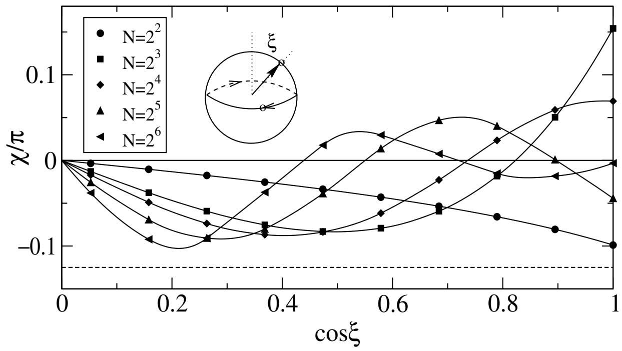

FIG. 1. The Berry phase χ for looping one MR quasihole around the equator with another quasihole fixed at a zenith angle ξ. $N = 4, 8, 16, 32, 64$ is the number of electrons. The dashed line, $\chi/\pi = -1/8$, shows a naive prediction. For $\cos\xi \approx 0$, the two quasiholes approach each other very closely and we see strong finite-size oscillations in the Berry phase. For larger N and $\cos\xi$ (i.e., larger quasihole separation in units of the magnetic length), χ appears to be converging toward zero. $\chi(-\cos\xi) = -\chi(\cos\xi)$.

Therefore, when another quasihole is moved inside the loop, the phase χ drops by $2\pi/p$ which implies fractional statistics of the quasiholes. In spherical geometry [8], the same result holds unless the south and north poles (which have singularities in our choice of gauge) are located on different sides of the loop. In the latter case, the Berry phase is given by $\chi = \pi\langle N_{\rm in} - N_{\rm out}\rangle$, where $N_{\rm in(out)}$ is the number of electrons inside (outside) the loop. If a single Laughlin quasihole is then looped around the equator, its Berry phase vanishes, but if another quasihole is placed above or below, the phase becomes $\chi = \pm\pi/p$. We check our Monte Carlo method by reproducing these results numerically. The charge of the MR ($\nu = 1/2$) quasihole is $e/4$, so that by analogy with the Laughlin state one might naively expect that the Berry phase for looping one quasihole around the equator with another fixed above or below it is given by $\chi = \pm\pi/8$ [11] (with an extra factor of $1/2$ due to MR quasiholes corresponding to only half of the flux quantum). In Fig. 1 we show the numerical calculation of χ for a MR system having two quasiholes, one looped around the equator and the other held fixed. If the two quasiholes approach each other too closely, we see strong finite-size oscillations in the Berry phase. However, for larger separation, χ appears to be converging towards zero, which was first predicted in Ref. [15] and can be well understood using the plasma analogy [16].

Even though the relative statistics of two MR quasiholes are trivial, they do pick up a phase due to their wrapping around the electrons, analogous to what occurs in the Laughlin case. Figure 2 shows that as the size of the system increases, the phase accumulated by interchanging two quasiholes (filled symbols) or braiding one around the other (open symbols) can be well approximated by assuming the wave function rotates by π for each enclosed electron (compare to 2π for the Laughlin state), when the poles are not separated by the loop (and the effect of the pole singularities is analogous to that in the Laughlin state). Even for systems consisting of only four electrons, this approximation stays quite good if we correct the average electron density for the charge pushed out by one localized quasihole (see the dashed lines in Fig. 2). This method of correcting the average density also works for the Laughlin state on the sphere.

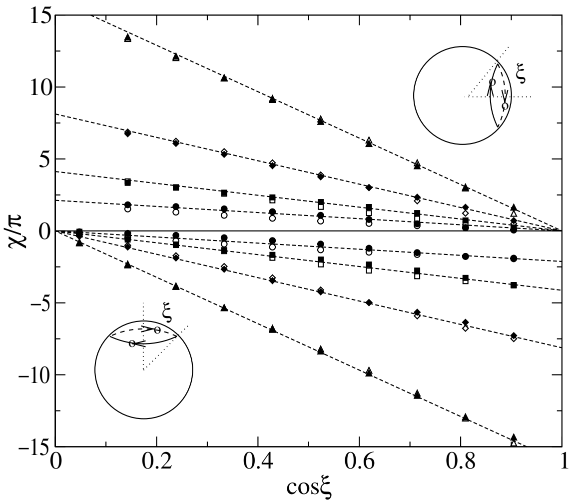

FIG. 2. For $\chi > 0$ ($\chi < 0$) filled symbols show the phase accumulated by interchanging two quasiholes around a circle with opening angle ξ centered on the equator (north pole), for various N as in Fig. 1. The straight dashed lines in the upper half are $0.5(N + 1/4)(1 - \cos\xi)$, corresponding to the expectation of the number of electrons enclosed by the loop. The $+1/4$ accounts for the charge pushed out by one of the quasiholes. For $\chi < 0$, the dashed lines are $-0.5(N + 1/4)\cos\xi$, i.e., one-half of the number of electrons inside minus one-half the number outside the loop. Open symbols, corresponding to a similar calculation with one quasihole moving and the other fixed at the center of the circle, almost overlay the filled symbols, confirming the trivial relative statistics.

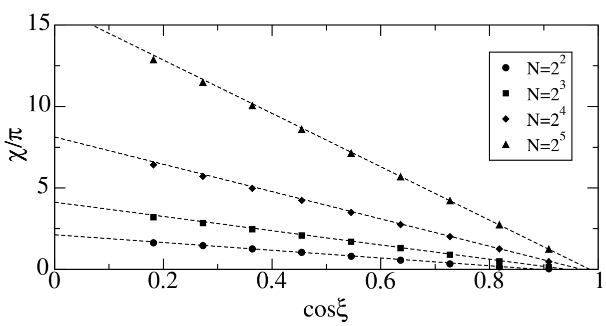

FIG. 3. Same as the upper half of Fig. 2, but now with four quasiholes present, two of which are fixed on the equator, at $\phi = \pm 3\pi/4$, and two interchanged, with initial and final positions at $\phi = \pm\xi$ on the equator. The straight dashed lines are $0.5(N + 3/4)\cos\xi - 1/4$. Here, $+3/4$ accounts for the average electron-density correction for the charge localized at $2n - 1$ quasiholes. The additional phase offset of $-1/4$ reflects the Abelian part of the braiding statistics, in agreement with the predictions of Refs. [3,10].

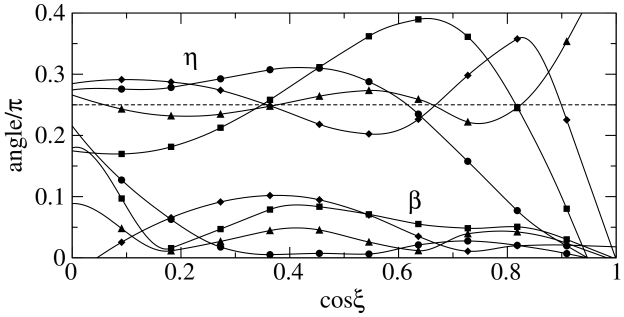

FIG. 4. Parameters η and β defining transformation matrix (5) for the same operations as χ shown in Fig. 3. The dashed line shows 1/4, an approximation used for η in the text. Similarly β can be approximated as zero [so that ϵ in Eq. (5) is not defined]. These approximations become better with larger system size and for intermediate $\cos\xi$ when the quasiholes remain farther apart. The symbol convention is the same as in Fig. 3. Lines interpolate Monte Carlo results.

We now turn to $2n = 4$ MR quasiholes, which is the simplest case when statistics can be non-Abelian (the ground state has degeneracy 2). While the above results for two quasiholes are anticipated by the plasma analogy [16], one may need deeper CFT [3,10] arguments in order to understand the following findings. In the calculation, we first fix all quasiholes on the equator and then interchange an adjacent pair of them around a circle with different opening angles ξ centered on the equator. Parametrizing a unitary matrix U by

$$U = e^{i\chi}\begin{pmatrix} e^{i\eta}\cos\beta/2 & ie^{-i\epsilon/2}\sin\beta/2 \\ ie^{i\epsilon/2}\sin\beta/2 & e^{-i\eta}\cos\beta/2 \end{pmatrix}, \tag{5}$$

we plot in Figs. 3 and 4 the results (in a convenient basis) for the transformation U_1 corresponding to the braiding operation on one of the quasihole pairs. Because of the rotational symmetry around the vertical axis, knowing U_1 we can deduce other transformations U_2, U_3, and U_4 (for interchanges of pairs ordered along the equator) by rotating and projecting the initial basis and correspondingly transforming U_1. It is then easy to show that $U_1 = U_3$ and $U_2 = U_4$ due to the form (2) of the wave function. Furthermore, we find numerically that $U_2 \approx F^\dagger U_1 F$, where $F = (\sigma_z - \sigma_x)/\sqrt{2}$, σ's being the usual Pauli matrices. This approximation is good within a few percent for smaller systems and is even better for larger ones.

According to Fig. 4, we see that apart from the Abelian phase χ, U_1 can be approximated by $U_1 \approx \mathrm{diag}(1 + i, 1 - i)/\sqrt{2}$, with the disagreement becoming smaller for larger systems. Using F, we can then construct all other matrices U_i. After performing the above approximations, we find that the unitary transformations corresponding to the braid operators realize the right-handed spinor representation of $SO(2n) \times U(1)$ (restricted to $\pi/2$ rotations around the axes) as predicted in Ref. [10] using CFT. In addition to the usual relations required of a representation of the braid group on the plane, on the sphere the generators must obey an additional relation. For the case of $2n = 4$, for example, we expect to have $U_1U_2U_3U_3U_2U_1 = 1$. One can easily show that (for general n) the relevant representation of the braid group predicted in Ref. [10] satisfies this additional relationship up to an Abelian phase. (The failure of the Abelian phase to satisfy this law is related to the gauge singularities and will be discussed elsewhere.)

In summary, we formulated a numerical method to study braiding statistics of FQH excitations and applied it to perform the first direct calculation of the non-Abelian statistics in the MR state. Our findings confirm results previously drawn within the CFT framework.

We have enjoyed helpful discussions with B. I. Halperin, C. Nayak, F. von Oppen, and N. Read. This work was supported in part by NSF Grant No. DMR 99-81283. The computations were performed on the O2000 supercomputer CPU farm at Boston University.

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