Parameter Scaling in the Decoherent Quantum-Classical Transition for Chaotic Systems

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The quantum to classical transition for a system depends on many parameters, including a scale length for its action, $\hbar$, a measure of its coupling to the environment, D, and, for chaotic systems, the classical Lyapunov exponent, λ. We propose measuring the proximity of quantum and classical evolutions as a multivariate function of $(\hbar, \lambda, D)$ and searching for transformations that collapse this hypersurface into a function of a composite parameter $\zeta = \hbar^{\alpha}\lambda^{\beta}D^{\gamma}$. We report results for the quantum Cat Map and Duffing oscillator, showing accurate scaling behavior over a wide parameter range, indicating that this may be used to construct universality classes for this transition.

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The classical description of a system approximates the inherently quantum world and has significantly different predictions. The question of when quantum mechanics reduces to classical behavior is both fundamentally interesting as well as relevant to applications such as quantum computing which seek to exploit this difference. The quantum to classical transition (QCT) is now understood to be affected not only by the relative size of $\hbar$ (Planck's constant) for a given system but also by D, a measure of the coupling of the environment to the quantum system of interest, an effect termed decoherence. Further, in systems where the classical evolution is chaotic, the transition is also influenced by the chaos, and thus by λ, the Lyapunov exponent of the classical trajectory dynamics [1,2]. Other work has addressed correspondence via the continuous extraction of information from the environment [3]. Here again, the condition for correspondence depends on both $\hbar$ and a parameter similar to D representing the measurement strength. As such, the QCT for general chaotic Hamiltonians is a complicated function of multiple parameters and is far from being fully understood.

However, progress has recently been made on various fronts. It has been argued that Hamiltonian systems fall into universality classes with distinctly different QCTs [4], behavior manifested in the density matrix far from the transition regime. Several studies also suggest that the behavior does not depend independently upon each of the three parameters. Specifically, considerations [1,2,5–8] of stochastic quantum evolution or a master equation argue that the classical limit, in particular, relates to a composite parameter involving $\hbar, D, \lambda$. With these and analytical arguments given below as motivation, we conjecture that crucial regions of the multiparameter QCT may, in fact, be collapsed via scaling relationships between these parameters. In particular, we propose (a) computing measures which directly reflect the "distance" between quantum and classical evolutions as a function of $\hbar$, λ, and D and then (b) searching for transformations that collapse the resulting hypersurface onto a function of a composite parameter of the form $\zeta = \hbar^{\alpha}\lambda^{\beta}D^{\gamma}$. The aims are (i) to search for this scaling, especially the coefficients α, β, γ [9], (ii) to investigate the range of parameters over which the scaling holds, and (iii) to study the dependence of the distance measure on ζ. This is expected to uncover universal behavior and considerably enhance our map of the quantum-classical boundary for nonlinear Hamiltonians.

We start with broad arguments for such scaling and then introduce measures of the quantum-classical distance including a generalized Kullback distance [10]. We then numerically test our ideas on the noisy quantum Cat Map. For this system, λ is a constant, such that the QCT is at most a two-parameter transition. We show that this transition, in fact, reduces to an effective single-parameter transition. The scaling is remarkably sharp and extends over a large parameter range. Therefore, in the Cat Map, the quantum nature of the system is a well-defined function of an effective Planck's constant, $\zeta \equiv \hbar^2\lambda D^{-1}$, consistent with previous analysis [7]. We discuss the nature of the transition in some detail. We then present results for a quantum Duffing oscillator coupled to the environment and show that its transition regime is remarkably similar to that of the Cat Map. This suggests the existence of a universality class for the QCT for completely chaotic systems. We conclude with possible expectations for the decoherent QCT in general nonlinear systems.

Consider a quantum Wigner quasiprobability ρ^W evolving under a Hamiltonian H with potential $V(q)$ while coupled to an external environment [1]:

$$\frac{\partial \rho^W}{\partial t} = \{H, \rho^W\} + \sum_{n\geq 1} \frac{\hbar^{2n}(-1)^n}{2^{2n}(2n+1)!}\frac{\partial^{2n+1}V(q)}{\partial q^{2n+1}}\frac{\partial^{2n+1}\rho^W}{\partial p^{2n+1}} + D\nabla^2\rho^W. \tag{1}$$

The first term on the right is the Poisson bracket, generating the classical evolution for ρ^W. The terms in $\hbar$ add the

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quantal evolution, while the effects of the environmental coupling are reflected in the diffusive term. For simplicity, we couple to all phase-space variables, although the results generalize. Consider for the moment only the classical evolution in the presence of the environmental perturbation. Chaos causes the density ρ to develop fine-scale structure exponentially rapidly, with a rate given by a generalized Lyapunov exponent. When the structure reaches sufficiently fine scales, the noise becomes important, acting to wipe out (or coarse-grain) fine-scale structure. The competition between chaos and noise is monitored by $\chi^2 \equiv \mathrm{Tr}[\rho^W \nabla^2 \rho^W]/\mathrm{Tr}[(\rho^W)^2] = -\mathrm{Tr}[|\nabla \rho^W|^2]/\mathrm{Tr}[(\rho^W)^2]$, obtained by an integration by parts and where Tr denotes the trace over all variables. This quantity χ^2 measures the structure in the distribution [11]. For systems with *both* classical chaos and noise, χ^2 arrives at the metastable value $\chi^{2*} = \sum_i \Lambda_{2,i}^+/2D \equiv \Lambda/2D$ where the $\Lambda_{2,i}^+$ are generalized positive Lyapunov exponents [10,12]. The quantum mechanical corrections, of the form $\hbar^{2n}[\partial^{2n+1}V(q)/\partial q^{2n+1}](\partial^{2n+1}\rho^W/\partial p^{2n+1})$, scale as $\hbar^{2n} \times \chi^{2n+1} V^{(2n+1)}(x)$, where $V^{(r)}$ denotes the rth derivative of V. Since the classical χ^2 for chaotic systems settles to the fixed value $\Lambda/2D$, the first difference between the quantum and classical evolution equations can be estimated to be $\zeta \equiv \hbar^{2n}\Lambda^{n+1/2}D^{-(n+1/2)}V^{(2n+1)}(x)$ where $x \approx \chi^{-1} = \sqrt{D/\Lambda}$. Therefore, quantum-classical distances should scale, in complete generality, with the single parameter ζ for small ζ. The particular form of ζ depends on the details of H and, in general, on the difference between the quantal and classical propagators.

As a measure of the distance between two distributions P and Q with support on the same space, we introduce

$$K_\epsilon(P, Q) = -\tfrac{1}{\epsilon}[\ln(\mathrm{Tr}[PQ^\epsilon]) - \ln(\mathrm{Tr}[P^{1+\epsilon}]) + \ln(\mathrm{Tr}[P^\epsilon Q]) - \ln(\mathrm{Tr}[Q^{1+\epsilon}])]. \quad (2)$$

K_ϵ is a generalized Kullback-Liebler (K-L) distance, which reduces to a symmetrized version of the usual K-L distance [10] in the limit $\epsilon \to 0$. K_ϵ has similar properties and is a general measure of the distance between the two probability distributions. When P and Q are identical, this measure is zero. A convenient form of K_ϵ is for $\epsilon = 1$ when it reduces to $K_1(P, Q) = \ln[\mathrm{Tr}[P^2]\mathrm{Tr}[Q^2]/(\mathrm{Tr}[PQ])^2]$.

We begin with an initial phase-space distribution ρ_0, which is propagated in time using separately (i) the quantum dynamics to yield $\rho_W(t)$ and (ii) the classical dynamics for $\rho_c(t)$. During the propagation, the distance $K_1(\rho_W, \rho_c)$ is monitored. $K_1(t = 0) = 0$, and due to diffusive noise all initial distributions relax to the constant distribution, such that $K_1(t \to \infty) = 0$, whence K_1 is a bounded function of time. For a given set of parameters $\hbar$, D and a reasonably long time $t_m (\gg 1/\Lambda)$, the maximal value of $K_1^m(\rho_W, \rho_c)$ is our measure of the quantum-classical distance.

We now consider a simple but extensively studied system, the noisy quantum Cat Map [2,7,13]. The classical limit displays extreme(uniformly hyperbolic) chaos, and as such the system is expected to belong to a distinct universality class. The uniform hyperbolicity also precludes any dependence on initial conditions. The dynamics derive from the kicked oscillator Hamiltonian [14]

$$H = p^2/2\mu + \epsilon q^2/2 \sum_{s=-\infty}^{\infty} \delta(s - t/T), \quad (3)$$

restricted to the torus $0 \le q < a$, $0 \le p < b$, with the parameter constraints $Tb/\mu a = 1$ and $-\epsilon Ta/b = 1$. The chaos here results not from the nonlinearity of the Hamiltonian but from the choice of (reinjected) boundary conditions. As such, the general equation Eq. (1) does not apply. However, the first quantum correction to the classical propagator for the Fourier-transformed distribution is of order $\hbar k$ for the Fourier mode k [13]. The quantum-classical distance for this system should then behave as $\hbar\chi$, implying that [15] $\zeta = \hbar^2\chi^2 = \hbar^2\Lambda D^{-1}$. The top panel of Fig. 1 shows K_1^m as a function of $\hbar$, D. The lower panel shows the same data as a function of the composite variable $\zeta = \hbar^2/D$. The reduction of the surface to a line demonstrates the scaling between $\hbar$, D, with the accuracy reflected in the lack of discernible spread around the curve. Remarkably, the scaling extends over many orders of magnitude in $\hbar$, D and a considerable range in K_1^m.

The graph of K_1^m versus ζ has a number of distinctive features: (i) K_1^m is monotonic in ζ, although as argued below, this need not be generally true. (ii) The distance K_1 is bounded due to the noise, as seen in the saturation as ζ increases. (iii) The quantum-classical distance is nonlinear in ζ, with $K_1^m(\zeta)$ initially growing slowly as a function of ζ, followed by a rapid transition at $\ln(\zeta) \approx 0$ or $\zeta \approx 1$, which is consistent with previous results [2,7,8]. These results indicate distinct regimes of small and large quantum-classical distance, understood as follows: in chaotic dynamics, a classical ρ develops fine-scaled structure very quickly (χ^2 grows rapidly), increasing its entropy production rate as well as its sensitivity to external noise. For this class of systems, in the regime $\zeta < 1$, a quantum ρ initially remains close to the classical, consequently increasing its entropy production rate and the rate at which it becomes a mixed state. Hence, quantum effects are suppressed by noise, and the quantum-classical distance remains small for all times. In this regime, the environment *minimizes* the quantum-classical difference. For $\zeta > 1$, the quantum ρ does not initially follow the classical ρ to finer scales and does not become sensitive to noise. It then remains far from classical even as the noise alters the classical system. Here, the environment *exaggerates* the differences between quantum and classical probability dynamics. As such, $\zeta \approx 1$ acts as a "quantum-classical boundary" separating qualitatively different behaviors.

Our scaling analysis is general and it should be visible in other measures of the quantum-classical distance. In Fig. 2 we show results for an alternate measure $D\chi^2$,

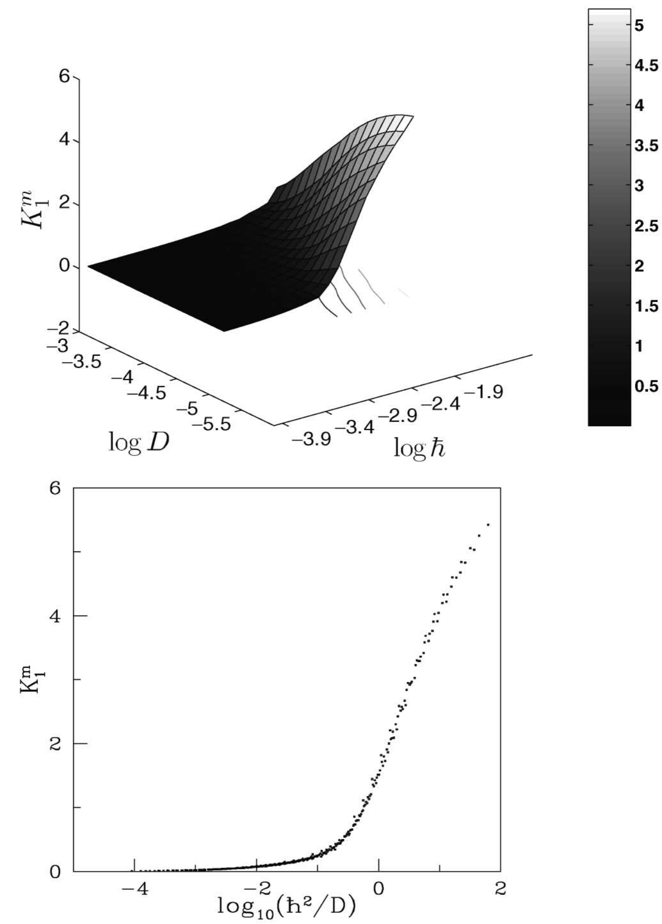

FIG. 1. Top: Maximal Kullback-Liebler distance K_1^m as a function of $\hbar$ and D, for the quantum Cat Map. Note that small values reflect strong similarity between classical and quantum evolutions. Bottom: Same data plotted in terms of a composite parameter reflecting scaling behavior.

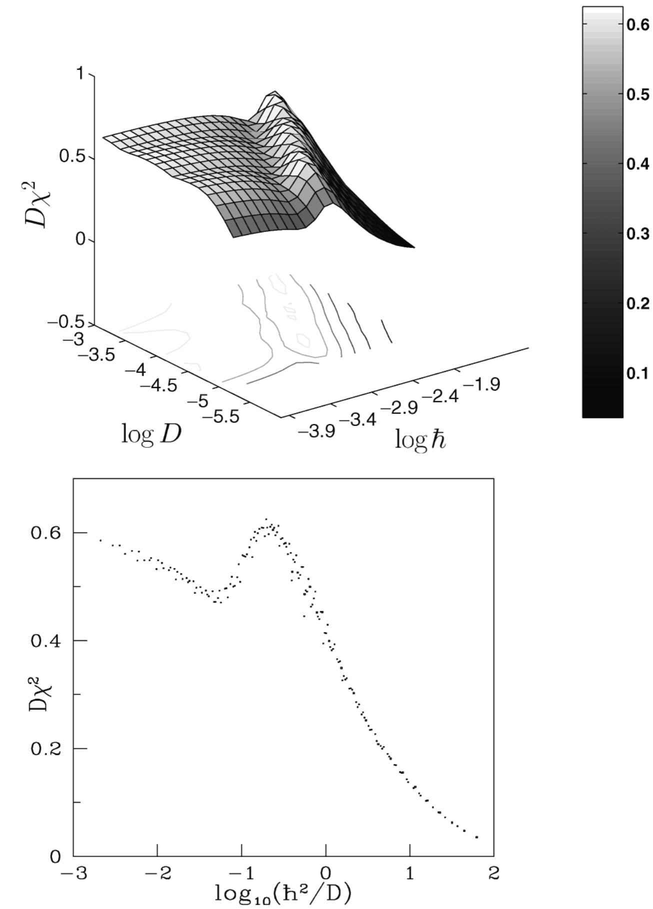

FIG. 2. Top: This measure reflects the generation of fine-scale structure in the dynamics with larger values corresponding to classical dynamics. Bottom: Same data plotted in terms of a composite parameter. Note the same scaling as in Fig. 1 and the coincidence of the transition region.

which is strictly quantum mechanical and is related to the spread of structure to finer scales. The supremum value in time of $D\chi^2$ ($\equiv D\chi_m^2$) is considered with varying $\hbar, D$ and for the same time scales as before (classically, we would get a constant [12]). Again, the precision and range of the scaling is remarkable. The qualitative conclusions are exactly the same as for $K_m^1(\zeta)$, with a similar rapid transition between large and small values of $D\chi_m^2$, happening again at $\zeta \approx 1$. However, this curve has a distinctive dip near $\zeta \approx 1$, such that the peak is at finite ζ. This has been seen previously [11] and can be understood by the fact that, for near-classical quantum dynamics, the quantum follows the classical distribution but carries interference fringes on top of the classical structure. As such, the quantum distribution can be more sensitive to noise than the classical counterpart. To see this, as above $\rho_W \approx \rho_c + a\hbar\chi\rho$ where a is some constant. Therefore, the quantum and classical χ^2 are related as $\chi_q^2 \approx \chi_c^2 + a\hbar\chi^3$ so that to zeroth order $\chi_{q0}^2 = \chi_c^2$, where the subscript on χ_q indicates the order. To first order, we substitute the zeroth order expression for χ^3 to get $\chi_{q1}^2 \approx \chi_c^2 + a\hbar\chi_c^3$. Iterating to second order we get terms such as [16] $\chi_{q2}^2 \approx \chi_c^2 + a\hbar\chi_c^3(1 + a\hbar\chi_c)^{3/2}$. For small $a\hbar$, this becomes

$$\begin{aligned}\chi_q^2 &\approx \chi_c^2(1 + a\hbar\chi_c + \tfrac{3}{2}a^2\hbar^2\chi_c^2 + \tfrac{3}{8}a^3\hbar^3\chi_c^3 + \ldots)\\ &\approx \chi_c^2(1 + a'\zeta^{1/2} + b\zeta + c\zeta^{3/2} + \ldots),\end{aligned} \quad (4)$$

where a', b, c absorb all other constants and we have substituted $\hbar^2\chi^2 = \zeta$. Since initially quantum dynamics reduces the value of χ^2, a' (and consequently c) must be negative valued constants, while b is positive. For appropriate values of a', b, c, Eq. (4) then indeed accounts for the shape of the curve seen in Fig. 2. Therefore, all measures of quantum-classical distance need not depend monotonically on the system parameters.

The smoothness and breadth of the scaling can be attributed to the uniform hyperbolicity of the Cat Map. However, the conclusions seem to transfer surprisingly well to a less formal system. We have studied the Duffing problem, with $V(x) = -ax^2 + bx^4 + cx\cos(\omega t)$. Here the symmetrized K-L distance was measured between the classical and quantal autocorrelation functions $\{\mathrm{Tr}[\rho(t)\rho(0)]\}$. In the deep chaotic regime where the phase

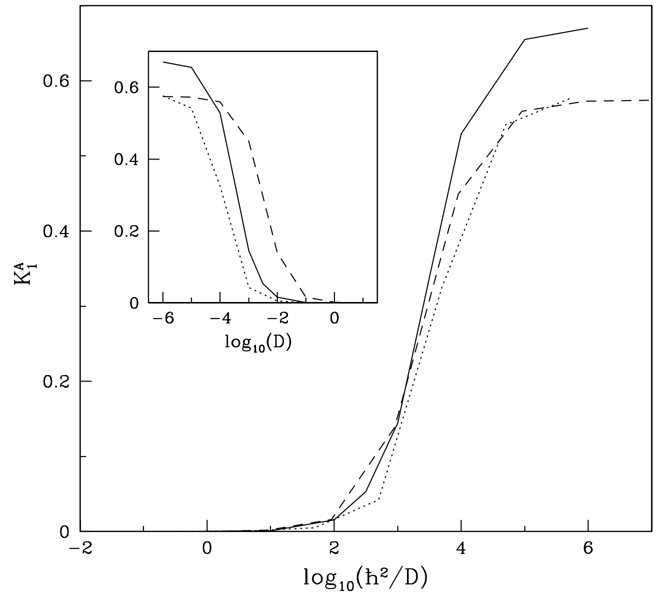

FIG. 3. K_1^A, the K-L distance computed between classical and quantal autocorrelation functions for the Duffing oscillator with $\hbar = 1$ (solid), $1/\sqrt{2}$ (dashed), 3 (dotted). The $\hbar^2/D$ scaling clearly brings the three disparate curves (inset) together. The Duffing parameters are $a = 10$, $b = 0.5$, $c = 10$, and $\omega = 6.07$.

space is almost entirely chaotic but not uniformly hyperbolic, scaling similar to the Cat Map is seen, as shown in Fig. 3. The only quantum correction term for the Duffing problem is precisely of the form to yield a $\hbar^2/D$ dependence for the scaling. For technical reasons, the initial conditions depend upon the choice of $\hbar$, which may be why the extreme quantum limit shows differing saturation points. Notice that there is an almost equally sharp QCT for this system as well, suggesting that the argument above about qualitatively different sensitivities to noise holds, independent of uniform hyperbolicity in the chaotic regime. Interestingly, such scaling is *not* seen for the Duffing in the integrable limit as, for example, when the driving is turned off. This confirms that the balance between chaos and noise leading to the metastable value of χ^2 is responsible for the scaling and is indirect evidence for the role of the Lyapunov exponent in this scaling. Moreover, the similarity in the QCT (including the sharp transition) for two very disparate systems—the first a formal system with chaos due to boundary conditions and the other with an analytic potential and nonuniform behavior—suggests a universality class for quantum-classical transitions.

The behavior of mixed phase systems, where chaotic and regular dynamics coexist, and of higher-dimensional systems now becomes a matter of interest that is being currently explored. We can anticipate two possible outcomes: first, that α, β, γ are independent of the Hamiltonian. This unlikely scenario is not supported by our analytical predictions but, were it to hold, a modified Planck's constant would govern all quantum chaotic systems, with possible universality classes arising from the dependence of a distance measure on ζ. Second is that systems show a range of behavior for α, β, γ, including a dependence on initial conditions. Specifically, the scaling may be related to the nature (single-scale, multiscale) of the quantum coherence affected by the environment. In this scenario, scaling would exist only for limited classes of systems, and the existence or range of scaling may be used to define universality classes.

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[15] We set $\zeta = \hbar^\alpha \Lambda^\beta D^\gamma$ and choose $\alpha = 2$ as discussed in [9] above. Since we show a logarithmic scale for ζ, this choice affects only the aspect ratio of the figure.
[16] In general, the expansion for χ_q around χ_c in terms of ζ includes terms in ζ arising from higher-order terms in the relationship between ρ_W and ρ_c. To establish that $\chi_q - \chi_c$ is nonlinear in ζ, these may be neglected since they add only to the nonlinearity.