Eigenvalues of the Zakharov-Shabat Scattering Problem for Real Symmetric Pulses

M. Desaix*
Department of Engineering Sciences, University College of Borås, SE-501 90 Borås, Sweden

D. Anderson, L. Helczynski, and M. Lisak
Department of Electromagnetics, Chalmers University of Technology, SE-412 96 Göteborg, Sweden
(Received 3 October 2002; published 7 January 2003)

The classical problem of determining the solitons generated from symmetric real initial conditions in the nonlinear Schrödinger equation is revisited. The corresponding Zakharov-Shabat scattering problem is solved for real and symmetric double-humped rectangular initial pulse forms. It is found that such real symmetric pulses may generate eigenvalues with nonzero real parts corresponding to separating soliton pulse pairs. Moreover, it is found that the classical formula relating the number of eigenvalues to the area of the pulse is not always correct.

DOI: 10.1103/PhysRevLett.90.013901 PACS numbers: 42.65.Tg, 42.65.Sf

The nonlinear Schrödinger (NLS) equation is one of the most fundamental nonlinear evolution equations in physics. An outstanding achievement in mathematical physics during the last 50 years is the development of the inverse scattering transform (IST) technique for solving the NLS equation for arbitrary initial conditions [1,2]. In the case of the NLS equation with a focusing nonlinearity, one of the most intriguing features of the solutions is the presence of stationary and very stable wave pulses—soliton pulses—emerging asymptotically for certain initial conditions. The problem of determining which initial pulse forms generate soliton pulses has attracted much interest. The answer to this fundamental question is contained in the eigenvalues of the Zakharov-Shabat scattering problem, which forms a crucial part of the IST. The real and imaginary parts of the eigenvalue determine the velocity and the amplitude, respectively, of the soliton. The soliton content of different initial pulse forms has been investigated in many previous studies and in general it has been found that the eigenvalues of symmetric and real pulses are purely imaginary (i.e., the generated soliton pulses have no velocity in the considered frame of reference) (cf. Ref. [2]) and that the number of soliton pulses is directly related to the area of the pulse (cf., e.g., Refs. [3,4]).

However, recently the suspicion has been growing that these results, which previously were considered as very fundamental, may not contain the full picture [5–8]. In fact, it has become clear that it is not enough that the initial pulse is real and symmetric in order to draw the traditional conclusions about the properties of the solitons, i.e., the number of solitons and their concomitant zero velocity.

As will be demonstrated in the present work, the previously accepted picture of soliton generation from real and symmetric initial pulses is not complete. By analyzing the Zakharov-Shabat eigenvalue problem for real double-humped symmetric box profiles we show that the conventional expression for the number of solitons is, in fact, not reliable for such profiles. First, in addition to eigenvalues which are purely imaginary, there may also exist a number of eigenvalues containing both real and imaginary parts corresponding to symmetrically separating solitons. Thus, the total number of solitons may be larger than that predicted by the classical integral condition. Second, we show that neither is this condition true in the restricted sense of determining the number of solitons corresponding to purely imaginary eigenvalues.

In the case of a focusing nonlinearity, the standard normalized form of the NLS equation is

$$i\frac{\partial\psi}{\partial t}+\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}+|\psi|^2\psi=0, \qquad \psi(0,x)=q(x). \tag{1}$$

In the inverse scattering transform method for the NLS equation, an important part of the analysis is the solution of the Zakharov-Shabat scattering problem, where the initial pulse, $q(x)$, plays the role of a scattering potential. The characteristic eigenvalue equation reads

$$\left\{\begin{array}{ll}\frac{dv_1}{dx}=-i\zeta v_1+q(x)v_2 & v_1\to\exp(-i\zeta x)\\ \frac{dv_2}{dx}=-q^*(x)v_1+i\zeta v_2 & v_2\to 0\end{array}\right\} \quad \text{as } x\to-\infty. \tag{2}$$

The asymptotic behavior of the eigenfunctions as $x\to\infty$ is given by $v_1\to a(\zeta)\exp(-i\zeta x)$ and $v_2\to b(\zeta)\exp(i\zeta x)$. The discrete eigenvalues, ζ_n, are solutions of the equation $a(\zeta)=0$ in the upper complex plane. The conventional picture is that the number of solitons in a real and positive pulse is given by

 © 2003 The American Physical Society

$$N = \left\lfloor \frac{1}{2} + \frac{1}{\pi}\int_{-\infty}^{\infty} q(x)dx \right\rfloor, \tag{3}$$

where $\lfloor x \rfloor$ denotes the integer part of the argument x. Moreover, for most real symmetric initial pulses considered so far, the eigenvalues are found to be purely imaginary. However, Klaus *et al.* [5] demonstrated that this is not a general truth by numerically giving a counterexample within the set of double-humped box potentials (compare Fig. 1). With a numerical trial and error approach they demonstrated that for a certain combination of amplitude and pulse distance, the discrete spectrum of the double-humped potential contained an eigenvalue with a real part.

Inspired by this result we consider the problem in more detail and show that indeed the situation is more complicated than the generally accepted picture.

Consider the class of real symmetric initial pulses, which consists of double-humped rectangular box profiles such that the amplitude is constant and equal to A in the intervals $x_1 < |x| < x_2$ and zero otherwise. The corresponding Zakharov-Shabat scattering problem for

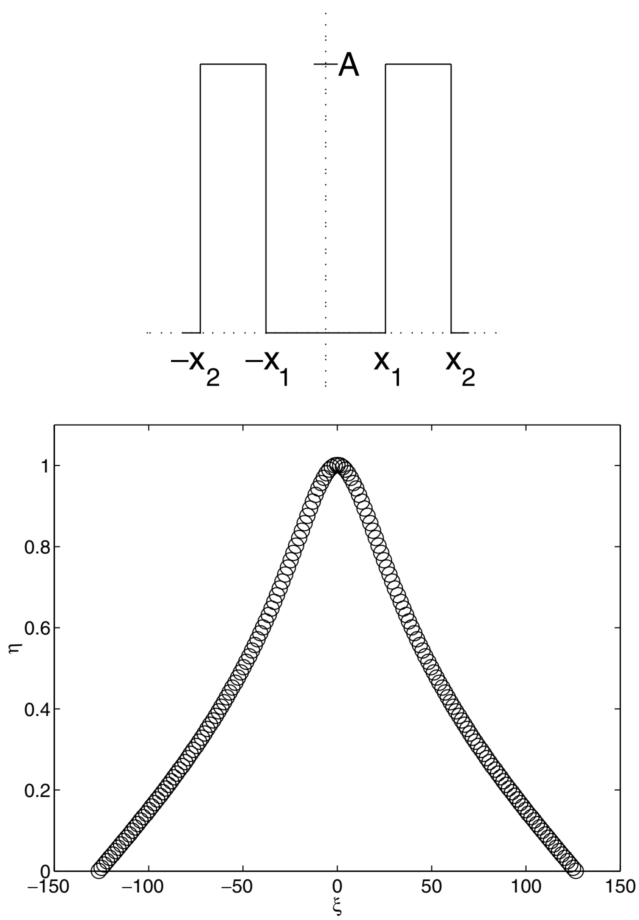

FIG. 1. Double-humped rectangular pulse and its corresponding eigenvalue distribution (open circles) in the complex plane ($A = 145.85$, $x_1 = 1.06$, and $x_2 = 1.07$).

pulses of this form can be solved by finding the solutions in the different regions where the potential is constant and then using the appropriate continuity conditions at the boundaries. This results in the following transcendental equation determining the eigenvalues ($\Delta \equiv x_2 - x_1$)

$$\cos^2(\Delta\sqrt{A^2+\zeta^2}) - \frac{A^2 e^{i4\zeta x_1} + \zeta^2}{A^2+\zeta^2}\sin^2(\Delta\sqrt{A^2+\zeta^2}) - \frac{i\zeta}{\sqrt{A^2+\zeta^2}}\sin(2\Delta\sqrt{A^2+\zeta^2}) = 0. \tag{4}$$

We emphasize that for $x_1 = 0$, Eq. (4) reduces to the standard eigenvalue equation for a rectangular pulse. For easy comparison with the numerical results obtained in Ref. [5], we use the amplitude $A = 145.85$. If $x_1 = 0$ and $x_2 = 0.01$, the initial condition has the form of a single-humped pulse. The corresponding transcendental equation has only one purely imaginary root in the upper complex plane, viz. $\zeta \approx i92.40$. The number of solitons, according to Eq. (3) is $N = \lfloor 1/2 + 2A\Delta/\pi \rfloor = \lfloor 1.43 \rfloor = 1$. Thus, for this case, the result is in accordance with the classical picture. However, if we consider a double-humped pulse with $x_1 = 1.06$ and $x_2 = 1.07$, the condition (3) is still the same and indeed only one purely imaginary eigenvalue is obtained ($\zeta \approx i1.00$). Nevertheless, the eigenvalue equation, Eq. (4), has a large number of eigenvalues with nonzero real part; compare Fig. 1. It is interesting to note that in the case of the double-humped profile, neither of the humps has enough energy to create a soliton on its own and consequently should start to decay as dispersive radiation. However, during the collision between the decaying radiation from the two humps, nonlinear effects are strong enough to create a large number of small solitons, one of them stationary, but all the others with symmetric and increasing velocities. The eigenvalues form an inverted V structure in the complex ζ plane, with the apex determined by the (single) imaginary eigenvalue, which also has the largest imaginary part. When the amplitude, A, of the box profile decreases, so does the number of eigenvalues, and for $A = A_{\mathrm{crit}} \approx 78.54$, the last (purely imaginary) eigenvalue disappears. We emphasize that this agrees with the threshold condition for soliton generation as predicted by Eq. (3).

Since the eigenvalues in the double-humped case analyzed here have real and imaginary parts that are much smaller than the initial amplitude, A, Eq. (4) can easily be solved approximately for the lowest order eigenvalues. When $|\zeta| \ll A$, the purely imaginary root, $\zeta = i\eta_0$, can be determined from

$$\cos^2(A\Delta) - \exp(-4x_1\eta_0)\sin^2(A\Delta) \approx 0. \tag{5}$$

This directly implies that $\eta_0 \approx \ln[\tan(A\Delta)]/(2x_1)$, which for the considered example yields $\eta \approx 1.03$, in good agreement with the numerical solution. For the neighboring pairs of complex eigenvalues, we can assume that the imaginary parts are close to that of the first purely imaginary eigenvalue, i.e., $\zeta \approx \xi + i\eta_0$. Using this fact and neglecting terms proportional to η/A, the eigenvalue equation reduces to simply

$$\exp(4ix_1\xi) \approx 1, \tag{6}$$

i.e., $\xi \approx \pm n\pi/(2x_1)$. The first pairs of complex eigenvalues ($\zeta_n = \xi_n + i\eta_0$) have $\xi_1 \approx \pm 1.48(1.44)$, $\xi_2 \approx \pm 2.96(2.89)$ and $\xi_3 \approx \pm 4.45(4.33)$, again in good agreement with the numerical solution (given in parentheses).

Since it has been demonstrated that Eq. (3) does not always correctly predict the number of solitons contained in a real symmetric initial condition, we recapitulate the derivation of Eq. (3) in order to understand why this is so. In the analysis of the threshold condition, it is assumed that $\zeta = 0$ so that the change of variable

$$z = \int_{-\infty}^{x} q(x')dx' \tag{7}$$

transforms the Zakharov-Shabat scattering equations to the harmonic oscillator equation. The boundary conditions $v_1 \to 1$ and $v_2 \to 0$, as $x \to -\infty$, determine the constants of integration, and the solution is given by

$$\begin{aligned} v_1 &= \cos\left(\int_{-\infty}^{x} q(x')dx'\right), \\ v_2 &= -\sin\left(\int_{-\infty}^{x} q(x')dx'\right). \end{aligned} \tag{8}$$

The condition for the number of discrete eigenvalues is assumed to follow from the condition $v_1 \to 0$ as $x \to \infty$, which reads

$$\int_{-\infty}^{\infty} q(x')dx' = \pi\left(n - \frac{1}{2}\right). \tag{9}$$

There are two major weak points in this analysis when it is claimed that Eq. (3) follows directly from Eq. (9). First, new eigenvalues of the form $\zeta = \pm\xi + 0i$ are inadvertently excluded from the analysis since the threshold condition is taken as $\zeta = 0 + 0i$. Consequently, the analysis is restricted to finding solitons born without velocity. Second, an amplitude determined by Eq. (9) may not necessarily correspond to the appearance of a new eigenvalue, it could equally well correspond to the disappearance of an old purely imaginary eigenvalue (cf. discussion below). This possibility makes the classical conclusion based on Eq. (3) premature.

An important aspect of the problem is to investigate under which conditions eigenvalues with real parts are born for the considered set of double-humped box profiles. For this purpose we start by using the fact that Eq. (4) can be factorized into two symmetric equations

$$\cot(\Delta\sqrt{A^2 + \zeta^2}) = \frac{i\zeta \pm Ae^{i2\zeta x_1}}{\sqrt{A^2 + \zeta^2}}. \tag{10}$$

When eigenvalues with real parts first appear, the eigenvalues must be of the form $\zeta = \pm\xi + 0i$. If Eq. (10) is separated into real and imaginary parts, the following two equations are obtained (cf. [6]):

$$\sin[A\Delta\sqrt{1 + (\xi/A)^2}] = \pm\sqrt{\frac{1 + (\xi/A)^2}{2}}, \tag{11}$$

$$\sin\left(2x_1A\frac{\xi}{A}\right) = \pm\frac{\xi}{A}. \tag{12}$$

This system contains three independent parameters, the relative magnitude of the eigenvalue, ξ/A, the normalized distance between the pulses, $2x_1A$, and the area of each pulse, $A\Delta$. We emphasize that the second of the relations directly determines the (normalized) real part of the eigenvalues, once the (normalized) distance between the pulses is given. However, the first relation provides an additional constraint for the existence of a solution. This can be viewed as a necessary condition on the area of the pulse for a given distance of separation between the subpulses. It is directly inferred from Eq. (12) that a necessary condition for solutions to exist is that $2Ax_1 > 1$; i.e., the normalized distance between the pulses must be larger than unity. Clearly, if $x_1 = 0$, i.e., the initial pulse is single-humped, no moving solitons can be generated. Thus the double-humped pulse form is a necessary (but not sufficient) condition for generation of moving solitons from real and symmetric initial conditions.

As a concluding example we consider the case of arbitrary amplitude A, but with each box of unity width ($\Delta = 1$) and a box separation of two units ($x_1 = 1$). From Eq. (9) we find that purely imaginary eigenvalues appear or disappear when

$$\int_{-\infty}^{\infty} q(x')dx' = 2A = \pi\left(n - \frac{1}{2}\right) \to A_c = \frac{\pi}{4}(2n - 1). \tag{13}$$

The critical amplitudes corresponding to the appearance of eigenvalues with real parts can be obtained by eliminating the amplitude between Eqs. (11) and (12), which yields the following transcendental equation for ξ:

$$\cot[\xi\sqrt{1 + \csc^2(2\xi)}] + \frac{\cot(2\xi)}{\sqrt{1 + \csc^2(2\xi)}} = 0, \tag{14}$$

which has an infinite number of roots, the two smallest being $\xi_1 = 0.937$ and $\xi_2 = 1.382$. The corresponding amplitudes can be found from Eq. (11) or Eq. (12) and are given by $A_1 = 0.982$ and $A_2 = 3.75$, respectively. Figure 2 shows a numerical solution of the Zakharov-Shabat scattering equations. From Eq. (13) we know that purely imaginary eigenvalues either appear or disappear when the amplitude is an odd multiple of $\pi/4$; in our case it turns out that a new eigenvalue appears at $A = \pi/4$ and still another at $A = 5\pi/4$, but that one eigenvalue disappears at $A = 3\pi/4$ in accordance with Eq. (13). The

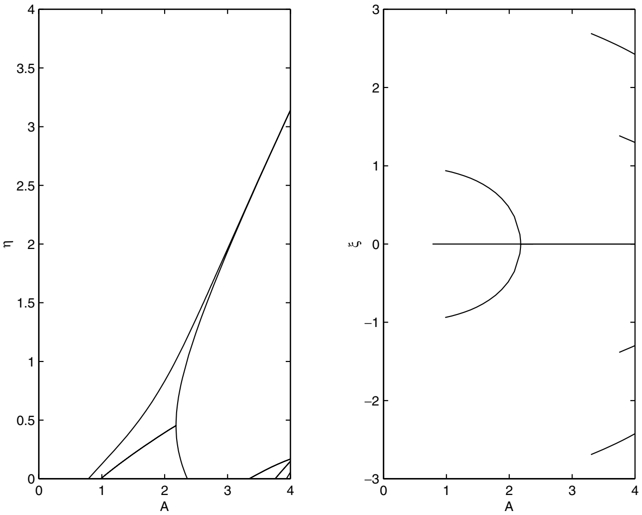

FIG. 2. Eigenvalues $\zeta = \xi + i\eta$ as a function of amplitude A for a double-humped potential with $\Delta = x_1 = 1$.

critical amplitudes for the appearance of eigenvalues with real parts are also seen to occur in accordance with the prediction above. The amplitude $A = 2$ corresponds to three eigenvalues, one purely imaginary and two symmetric with real parts, while for $A = 2.5$ only two purely imaginary eigenvalues are present. In Figs. 3 and 4 we have illustrated this by solving the nonlinear Schrödinger equation for the cases $A = 2$ and $A = 2.5$, respectively. Figure 3 shows that in the first case three solitons, one stationary and two with sign reversed velocities, are present. In the second case, Fig. 4, only a stationary two-soliton exists exhibiting the classical periodic collapse behavior.

In conclusion, we have analyzed the soliton content in a class of real and symmetric double-humped initial pulses. It is shown that, contrary to the accepted picture in this

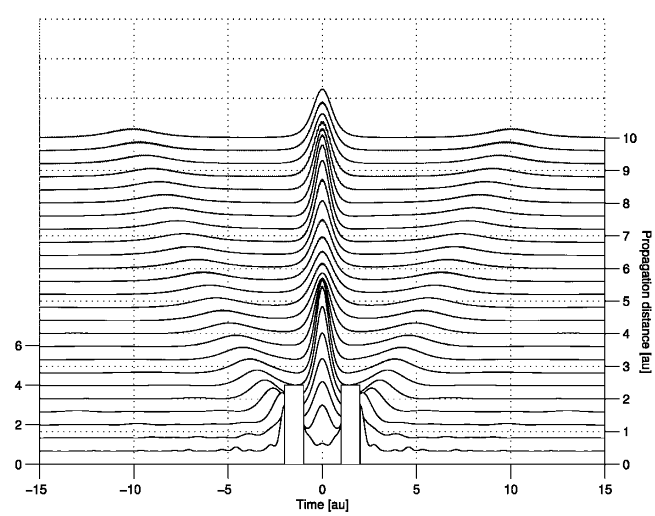

FIG. 3. Numerical solution of the nonlinear Schrödinger equation for a double-humped initial condition with $A = 2$ and $\Delta = x_1 = 1$.

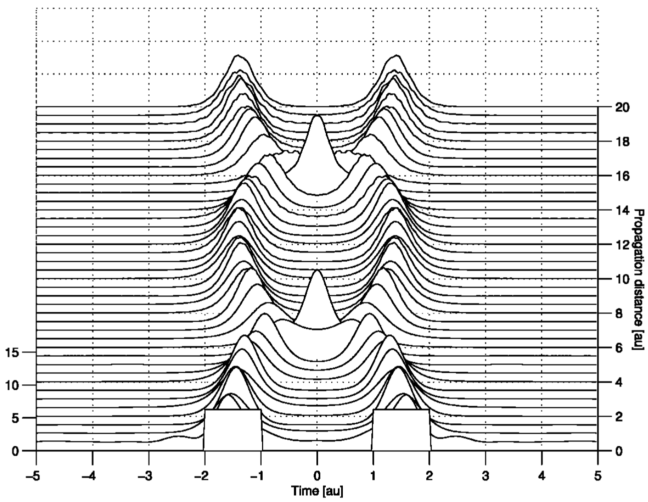

FIG. 4. Numerical solution of the nonlinear Schrödinger equation for a double-humped initial condition with $A = 2.5$ and $\Delta = x_1 = 1$.

situation, the corresponding Zakharov-Shabat scattering problem may have solutions for which the eigenvalues have real as well as imaginary parts, corresponding to separating soliton pairs. A closer investigation of the classical threshold for soliton generation from real and symmetric initial pulse conditions clearly identifies the weak points in the analysis and establishes a new understanding of the threshold condition. Several examples are investigated numerically to illustrate and support the analysis.

At the conference on Nonlinear Guide Waves and Their Applications in Stresa, Italy [7,8], we had the opportunity to discuss the present problem with F. Abdullaev. We gratefully acknowledge these inspiring discussions. We also thank E. N. Tsoy for sending us Ref. [6], which played an important part in the concluding stage of this work.

*Electronic address: Mats.Desaix@hb.se

[1] V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP **34**, 62 (1974).
[2] J. Satsuma and N. Yajima, Prog. Theor. Phys. Suppl. **55**, 284 (1974).
[3] Yu. S. Kivshar, J. Phys. A **22**, 337 (1989).
[4] M. Desaix, D. Anderson, M. Lisak, and M. L. Quiroga-Teixeiro, Phys. Lett. A **212**, 332 (1996).
[5] M. Klaus and J. K. Shaw, Opt. Commun. **197**, 491 (2001).
[6] F. Kh. Abdullaev and E. N. Tsoy, Physica (Amsterdam) **161D**, 67 (2002).
[7] M. Desaix, D. Anderson, L. Helczynski, and M. Lisak, in *Proceedings of the Conference on Nonlinear Guided Waves and Their Applications, Stresa, Italy, 2002* (Optical Society of America, Washington, DC, 2002).
[8] E. N. Tsoy and F. Kh. Abdullaev, in *Proceedings of the Conference on Nonlinear Guided Waves and Their Applications, Stresa, Italy, 2002* (Ref. [7]).