Two-Loop Renormalization Group Equations in the Standard Model

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Two-loop renormalization group equations in the standard model are recalculated. A new coefficient is found in the β function of the quartic coupling and a class of gauge invariants is found to be absent in the β functions of hadronic Yukawa couplings. The two-loop β function of the Higgs mass parameter is presented in complete form.

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Analysis based upon renormalization group equations (RGEs) plays an important role in the study of physics of the standard model (SM) and beyond. Detailed analysis of RGEs confirmed the behavior of asymptotic freedom in QCD, and thus helped to establish a non-Abelian gauge theory for the strong interaction [1]. The runnings of coupling constants and mass parameters are crucial in global analysis of high precision electroweak experiments [2]. On the other hand, RGEs analysis extrapolated to extremely high energy provides a possible test for physics beyond the SM. For example, gauge couplings do not unify within the SM. This gives extra evidence against simple grand unification theories such as SU(5) without supersymmetry, in addition to the nonobservation of proton decay. On the other hand, gauge couplings seem to unify at a scale $\sim 2\times 10^{16}$ GeV in the minimal supersymmetric standard model, which can be interpreted as an indirect evidence for supersymmetry as well as unification theories [3–5]. Comprehensive analysis can be found in [6].

Computations of RGEs in gauge theories have been performed for various models to different orders of perturbation. Persistent efforts yielded recently a four-loop result of the β function of the strong coupling constant [7]. Two-loop RGEs of dimensionless couplings in a general gauge theory as well as the specific case of the SM had been calculated long ago in a series of classic papers by Machacek and Vaughn [8–10]. By introducing a nonpropagating gauge-singlet "dummy" scalar field, two-loop RGEs of dimensional couplings can be readily inferred from dimensionless results [11,12]. These were used to derive the RGEs of supersymmetric theories a decade later [11].

In this paper, we recalculate the two-loop RGEs in the SM, in a combination of using the general results of [8–10] and direct calculations from Feynman diagrams. A new coefficient is found in the β function of the quartic coupling, and a class of gauge invariants are found to be absent in β functions of hadronic Yukawa couplings. We will also present the two-loop β function of the Higgs mass parameter in complete form, which provides a partial but useful check on the calculation of the quartic coupling. Whenever discrepancy with the literature appears, we carefully inspect relevant Feynman diagrams to ensure consistency.

To fix notations, we define Yukawa couplings and the Higgs potential in the SM to be

$$-\mathcal{L}_{\rm int} = \{\bar{e}\mathbf{F}_L\phi^+ l + \bar{d}\mathbf{F}_D\phi^+ q + \bar{u}\mathbf{H}\phi^{+c} q + \text{H.c.}\} + m^2\phi^+\phi + \frac{\lambda}{2}(\phi^+\phi)^2, \tag{1}$$

where three families of fermions are grouped together so $\mathbf{F}_L, \mathbf{F}_D, \mathbf{H}$ are 3×3 complex matrices, and $\phi^c \equiv i\tau_2\phi^*$. For each coupling constant x in Eq. (1), we define a corresponding β function,

$$\beta_x = \mu\frac{dx}{d\mu} = \frac{1}{16\pi^2}\beta_x^{(1)} + \frac{1}{(16\pi^2)^2}\beta_x^{(2)}, \tag{2}$$

where $\beta_x^{(1)}, \beta_x^{(2)}$ denote the one-loop and two-loop contributions, respectively. We use dimensional regularization and the modified minimal subtraction scheme ($\overline{MS}$) for renormalization. The expressions of the $\beta_x^{(1)}$ are quite standard which can be easily reproduced. The evaluation of $\beta_x^{(2)}$ will be the object of this article.

Following the conventions of [8–10], we define the following combinations of Yukawa matrices for later convenience:

$$Y_2(S) = \mathrm{Tr}[3\mathbf{H}^+\mathbf{H} + 3\mathbf{F}_D^+\mathbf{F}_D + \mathbf{F}_L^+\mathbf{F}_L],$$

$$H(S) = \mathrm{Tr}[3(\mathbf{H}^+\mathbf{H})^2 + 3(\mathbf{F}_D^+\mathbf{F}_D)^2 + (\mathbf{F}_L^+\mathbf{F}_L)^2],$$

$$Y_4(S) = \left(\frac{17}{20}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2\right)\mathrm{Tr}(\mathbf{H}^+\mathbf{H}) + \left(\frac{1}{4}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2\right)\mathrm{Tr}(\mathbf{F}_D^+\mathbf{F}_D) + \frac{3}{4}(g_1^2 + g_2^2)\,\mathrm{Tr}(\mathbf{F}_L^+\mathbf{F}_L),$$

$$\chi_4(S) = \frac{9}{4}\mathrm{Tr}[3(\mathbf{H}^+\mathbf{H})^2 + 3(\mathbf{F}_D^+\mathbf{F}_D)^2 + (\mathbf{F}_L^+\mathbf{F}_L)^2 - \frac{1}{3}\{\mathbf{H}^+\mathbf{H}, \mathbf{F}_D^+\mathbf{F}_D\}].$$

The complex Higgs doublet has to be decomposed into real fields. Further complication arises since [8–10] assumed implicitly that the fermion fields are real, while usual Weyl fermions are complex. Caution should be taken when Yukawa couplings and gauge representation matrices of fermions are dealt with [12]. All issues taken into account, β functions in the SM can be obtained in a straightforward manner. The lengthy algebra is greatly simplified with the aid of the symbolic software FORM [13]. The β functions of the gauge coupling constants are readily reproduced and conform to those in the literature[8]. First, we present the β functions of the Yukawa couplings. To one loop,

$$\mathbf{H}^{-1}\beta_H^{(1)} = \frac{3}{2}(\mathbf{H}^+\mathbf{H} - \mathbf{F}_D^+\mathbf{F}_D) + Y_2(S) - \left(\frac{17}{20}g_1^2 + \frac{9}{4}g_2^2 + 8g_3^2\right), \tag{3}$$

$$\mathbf{F}_D^{-1}\beta_{F_D}^{(1)} = \tfrac{3}{2}(\mathbf{F}_D^+\mathbf{F}_D - \mathbf{H}^+\mathbf{H}) + Y_2(S) - (\tfrac{1}{4}g_1^2 + \tfrac{9}{4}g_2^2 + 8g_3^2), \tag{4}$$

$$\mathbf{F}_L^{-1}\beta_{F_L}^{(1)} = \tfrac{3}{2}\mathbf{F}_L^+\mathbf{F}_L + Y_2(S) - \tfrac{9}{4}(g_1^2 + g_2^2); \tag{5}$$

to two loops,

$$\mathbf{H}^{-1}\beta_H^{(2)} = \frac{3}{2}(\mathbf{H}^+\mathbf{H})^2 - \mathbf{H}^+\mathbf{H}\mathbf{F}_D^+\mathbf{F}_D - \frac{1}{4}\mathbf{F}_D^+\mathbf{F}_D\mathbf{H}^+\mathbf{H} + \frac{11}{4}(\mathbf{F}_D^+\mathbf{F}_D)^2 + Y_2(S)\left(\frac{5}{4}\mathbf{F}_D^+\mathbf{F}_D - \frac{9}{4}\mathbf{H}^+\mathbf{H}\right) - \chi_4(S) + \frac{3}{2}\lambda^2 - 6\lambda\mathbf{H}^+\mathbf{H} + \frac{5}{2}Y_4(S) + \left(\frac{223}{80}g_1^2 + \frac{135}{16}g_2^2 + 16g_3^2\right)\mathbf{H}^+\mathbf{H} - \left(\frac{43}{80}g_1^2 - \frac{9}{16}g_2^2 + 16g_3^2\right)\mathbf{F}_D^+\mathbf{F}_D + \left(\frac{9}{200} + \frac{29}{45}n_g\right)g_1^4 - \frac{9}{20}g_1^2g_2^2 + \frac{19}{15}g_1^2g_3^2 - \left(\frac{35}{4} - n_g\right)g_2^4 + 9g_2^2g_3^2 - \left(\frac{404}{3} - \frac{80}{9}n_g\right)g_3^4, \tag{6}$$

$$\mathbf{F}_D^{-1}\beta_{F_D}^{(2)} = \frac{3}{2}(\mathbf{F}_D^+\mathbf{F}_D)^2 - \mathbf{F}_D^+\mathbf{F}_D\mathbf{H}^+\mathbf{H} - \frac{1}{4}\mathbf{H}^+\mathbf{H}\mathbf{F}_D^+\mathbf{F}_D + \frac{11}{4}(\mathbf{H}^+\mathbf{H})^2 + Y_2(S)\left(\frac{5}{4}\mathbf{H}^+\mathbf{H} - \frac{9}{4}\mathbf{F}_D^+\mathbf{F}_D\right) - \chi_4(S) + \frac{3}{2}\lambda^2 - 6\lambda\mathbf{F}_D^+\mathbf{F}_D + \frac{5}{2}Y_4(S) + \left(\frac{187}{80}g_1^2 + \frac{135}{16}g_2^2 + 16g_3^2\right)\mathbf{F}_D^+\mathbf{F}_D - \left(\frac{79}{80}g_1^2 - \frac{9}{16}g_2^2 + 16g_3^2\right)\mathbf{H}^+\mathbf{H} - \left(\frac{29}{200} + \frac{1}{45}n_g\right)g_1^4 - \frac{27}{20}g_1^2g_2^2 + \frac{31}{15}g_1^2g_3^2 - \left(\frac{35}{4} - n_g\right)g_2^4 + 9g_2^2g_3^2 - \left(\frac{404}{3} - \frac{80}{9}n_g\right)g_3^4, \tag{7}$$

$$\mathbf{F}_L^{-1}\beta_{F_L}^{(2)} = \frac{3}{2}(\mathbf{F}_L^+\mathbf{F}_L)^2 - \frac{9}{4}Y_2(S)\mathbf{F}_L^+\mathbf{F}_L - \chi_4(S) + \frac{3}{2}\lambda^2 - 6\lambda\mathbf{F}_L^+\mathbf{F}_L + \left(\frac{387}{80}g_1^2 + \frac{135}{16}g_2^2\right)\mathbf{F}_L^+\mathbf{F}_L + \frac{5}{2}Y_4(S) + \left(\frac{51}{200} + \frac{11}{5}n_g\right)g_1^4 + \frac{27}{20}g_1^2g_2^2 - \left(\frac{35}{4} - n_g\right)g_2^4, \tag{8}$$

where $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ gauge coupling constants g_3, g_2, and g_1 are normalized based upon SU(5), so the standard electroweak gauge coupling constants g and g' are related to these by $g^2 = g_2^2$ and $g'^2 = 3/5g_1^2$. The matrices $\mathbf{F}_L$, $\mathbf{F}_D$, and $\mathbf{H}$ do not have to be invertible. Their inverses should only be understood symbolically and need not be introduced in principle. Properly interpreted, the β functions are equal to the Yukawa matrices themselves multiplied by the right-hand side of the corresponding equations. For $\beta_H^{(2)}$ in [9], there was the term $-2\lambda\mathbf{H}\mathbf{F}_D^+\mathbf{F}_D$, which is absent in Eq. (6). A close inspection indicates that this term arises only from the Feynman diagrams shown in Figs. 1(a) and 1(b). However, these two diagrams cancel with each other, which can easily be verified by an elementary calculation. Similarly in Eq. (7) for $\beta_{F_D}^{(2)}$, the term $-2\lambda\mathbf{F}_D\mathbf{H}^+\mathbf{H}$ is also absent, in contrast with [9]. The corresponding Feynman diagrams are shown in Figs. 1(c) and 1(d), and again they cancel with each other. The leptonic results are included here for completeness.

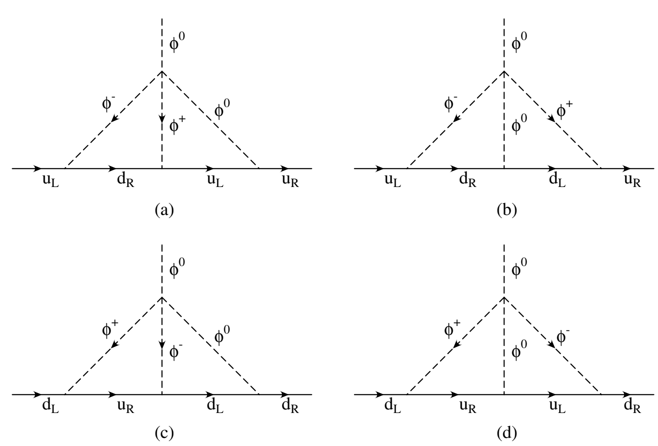

FIG. 1. Two-loop diagrams which affect hadronic Yukawa couplings. (a) and (b) [(c) and (d)] cancel with each other, thus resulting in a null contribution to $\beta_H^{(2)}$ ($\beta_{F_D}^{(2)}$).

The β function of the quartic coupling can be obtained in a similar manner. The calculation is greatly simplified by the fact that there is only one independent quartic coupling in the SM. Calculation for models beyond the SM with numerous quartic couplings would be more involved, due to proliferation of combinatorics. The one-loop contribution to the β function of λ is

$$\beta_\lambda^{(1)} = 12\lambda^2 - \left(\frac{9}{5}g_1^2 + 9g_2^2\right)\lambda + \left(\frac{27}{100}g_1^4 + \frac{9}{10}g_1^2g_2^2 + \frac{9}{4}g_2^4\right) + 4Y_2(S)\lambda - 4H(S); \tag{9}$$

to two loops,

$$\begin{aligned}\beta_\lambda^{(2)} = & -78\lambda^3 + \left(54g_2^2 + \frac{54}{5}g_1^2\right)\lambda^2 - \left[\left(\frac{313}{8} - 10n_g\right)g_2^4 - \frac{117}{20}g_2^2g_1^2 - \left(\frac{687}{200} + 2n_g\right)g_1^4\right]\lambda + \left(\frac{497}{8} - 8n_g\right)g_2^6 \\ & - \left(\frac{97}{40} + \frac{8}{5}n_g\right)g_2^4g_1^2 - \left(\frac{717}{200} + \frac{8}{5}n_g\right)g_2^2g_1^4 - \left(\frac{531}{1000} + \frac{24}{25}n_g\right)g_1^6 - 64g_3^2\,\mathrm{Tr}[(\mathbf{H}^+\mathbf{H})^2 + \mathbf{F}_D^+\mathbf{F}_D)^2] \\ & - \frac{8}{5}g_1^2\,\mathrm{Tr}[2(\mathbf{H}^+\mathbf{H})^2 - (\mathbf{F}_D^+\mathbf{F}_D)^2 + 3(\mathbf{F}_L^+\mathbf{F}_L)^2] - \frac{3}{2}g_2^4Y_2(S) + g_1^2\left[\left(\frac{63}{5}g_2^2 - \frac{171}{50}g_1^2\right)\mathrm{Tr}(\mathbf{H}^+\mathbf{H}) \right. \\ & \left. + \left(\frac{27}{5}g_2^2 + \frac{9}{10}g_1^2\right)\mathrm{Tr}(\mathbf{F}_D^+\mathbf{F}_D) + \left(\frac{33}{5}g_2^2 - \frac{9}{2}g_1^2\right)\mathrm{Tr}(\mathbf{F}_L^+\mathbf{F}_L)\right] + 10\lambda Y_4(S) - 24\lambda^2Y_2(S) - \lambda H(S) \\ & - 42\lambda\mathrm{Tr}(\mathbf{H}^+\mathbf{H}\mathbf{F}_D^+\mathbf{F}_D) + 20\,\mathrm{Tr}[3(\mathbf{H}^+\mathbf{H})^3 + 3(\mathbf{F}_D^+\mathbf{F}_D)^3 + (\mathbf{F}_L^+\mathbf{F}_L)^3] - 12\,\mathrm{Tr}[\mathbf{H}^+\mathbf{H}(\mathbf{H}^+\mathbf{H} + \mathbf{F}_D^+\mathbf{F}_D)\mathbf{F}_D^+\mathbf{F}_D]. \end{aligned} \tag{10}$$

Note in Eq. (10), the coefficient of the term $\lambda\,\mathrm{Tr}(\mathbf{H}^+\mathbf{H}\mathbf{F}_D^+\mathbf{F}_D)$ is -42, instead of 6 as given by [10]. We note that terms proportional to $\lambda H(S)$ and $\lambda\,\mathrm{Tr}(\mathbf{H}^+\mathbf{H}\mathbf{F}_D^+\mathbf{F}_D)$ arise partly from scalar boson propagators. The relevant Feynman diagrams are shown in Figs. 2(a) and 2(b). There are also related two-loop proper scalar quartic vertex diagrams, which are generically shown in Figs. 2(c) and 2(d). It turns out that Fig. 2(c) does not contribute to the β function, so only Figs. 2(a), 2(b), and 2(d) need to be evaluated. In addition to calculating Feynman diagrams directly, we compare the coefficient of $\lambda H(S)$ and that of $\lambda\,\mathrm{Tr}(\mathbf{H}^+\mathbf{H}\mathbf{F}_D^+\mathbf{F}_D)$. By including all specific diagrams and carefully collecting all coefficients, we find that the ratio of the term proportional to the latter over the term proportional to the former is 42, instead of -6. This substantiates Eq. (10). On the other hand, coefficients of terms $\lambda g_1^2g_2^2$ and λg_1^4 in Eq. (10) conform to those in [14], and in fact Eq. (10) agrees with the corresponding result in [14] if only the top-quark Yukawa coupling is retained.

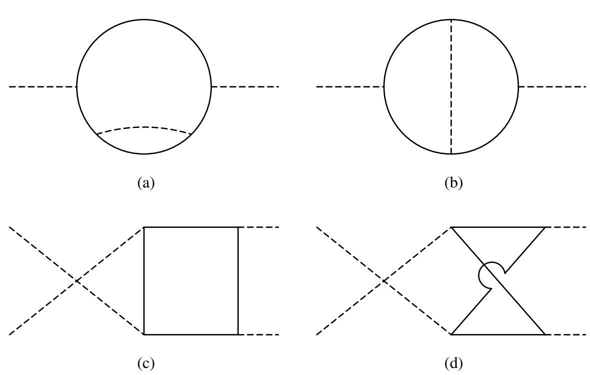

FIG. 2. (a) and (b): Part of hadronic Yukawa coupling contribution to Higgs boson propagators, which in turn affect β_λ; (c) and (d): relevant proper scalar quartic vertex diagrams.

The β function of m^2 can be inferred from the results in [10] by introducing a nonpropagating dummy real scalar field ϕ_d with no gauge interactions, and carefully computing the combinatorics associated with the symmetry factor [11,12]. Specifically, the mass term can be rewritten as

$$-\mathcal{L} = m^2\phi^+\phi = \tfrac{1}{4!}\lambda_{ddij}\phi_d\phi_d\phi_i\phi_j,$$

by decomposing the complex doublet ϕ into four real scalars ϕ_i, $(i = 1, 4)$. If ϕ_d is taken to have no other interactions, then the β functions of m^2 have the same form as that of the newly introduced quartic coupling λ_{ddij}. To one loop, the β function is

$$\beta_{m^2}^{(1)} = m^2\left[6\lambda + 2Y_2(S) - \frac{9}{10}g_1^2 - \frac{9}{2}g_2^2\right], \tag{11}$$

and to two loops,

$$\begin{aligned}\beta_{m^2}^{(2)} = m^2\Big[& -15\lambda^2 - 12\lambda Y_2(S) \\ & -\frac{9}{2}H(S) - 21\,\mathrm{Tr}(\mathbf{H}^+\mathbf{H}\mathbf{F}_D^+\mathbf{F}_D) \\ & + \left(\frac{36}{5}g_1^2 + 36g_2^2\right)\lambda + 5Y_4(S) + \left(n_g + \frac{471}{400}\right)g_1^4 \\ & + \frac{9}{8}g_1^2g_2^2 + \left(5n_g - \frac{385}{16}\right)g_2^4\Big]. \end{aligned} \tag{12}$$

Note the terms $n_g g_1^4$ and $n_g g_2^4$ in Eq. (12), which originate from generic Feynman diagrams shown in Fig. 3. These diagrams are proportional to the fermion Dynkin indices

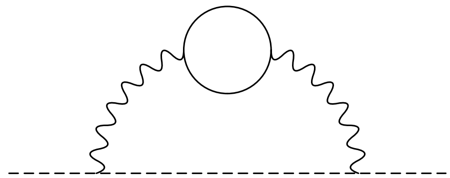

FIG. 3. Part of $SU(2) \times U(1)$ contributions to the Higgs boson propagators.

in the pure fermionic loops, while Dynkin indices are always positive. So all fermions in the loop contribute additively, thus resulting in the n_g factor in the expression. Equation (12) is consistent with the newly revised result of [14].

To a good approximation, we can express the SM effective potential of the Higgs field,

$$V_{\rm eff}(\phi) = \bar{m}^2(t)Z^2(t)\phi^+\phi + \tfrac{1}{2}\bar{\lambda}(t)Z^4(t)(\phi^+\phi)^2, \quad (13)$$

in terms of the running coupling constants and the running Higgs mass,

$$\frac{d\ln Z(t)}{dt} = -\gamma_\phi[\bar{x}(t)], \qquad \frac{d\bar{m}^2(t)}{dt} = \beta_{m^2}[\bar{x}(t), \bar{m}^2(t)],$$
$$\frac{d\bar{x}(t)}{dt} = \beta_x[\bar{x}(t)], \quad (14)$$

where the generic symbol x represents all dimensionless couplings, including the gauge couplings, the Yukawa couplings, and the quartic scalar coupling; γ_ϕ is the anomalous dimension of the Higgs field, to be found in [8,12]; $t = \log\phi/M$, where M is an arbitrary mass scale, at which the initial values of the coupling constants and the mass are defined. The vacuum expectation value of the Higgs field is determined by minimization of $V_{\rm eff}$. The physical mass of the Higgs boson is simply the second derivative of $V_{\rm eff}$ evaluated at the minimum [15].

In summary, we have recalculated the RGEs in the SM. A new coefficient is found in the β function of the quartic coupling and a class of gauge invariants are found to be absent in β functions of hadronic Yukawa couplings. The β function of the Higgs mass parameter is also presented in complete form. The changes in Yukawa couplings affect the running of the Cabibbo-Kobayashi-Maskawa matrix and the quark masses. The changes in m^2 and λ will change the Higgs potential, which in turn affect the triviality and vacuum stability bound on the Higgs mass. Because of the dominance of one-loop results and relative bigger contributions from gauge couplings over those from the quartic and Yukawa couplings (with the exception of the top quark), numerical changes are not expected to be significant. For λ and Yukawa couplings related to the b quark, the changes are magnified by the factor m_t^2/M_W^2, but again suppressed by the factor m_b^2/M_W^2. All these shifts will be included in a future comprehensive analysis [12]. On the other hand, for a heavy fourth family of fermions, the changes would be sizable.

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Note added.—Upon completion of this work, we were informed that part of the results presented here was also reached in [16].

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