[6]Appendix of M. Sugawara and A. Kanazawa, Phys. Rev. 123, 1895 (1961).

[7]Appendix 2 of M. Sugawara and A. Kanazawa, Phys. Rev. 126, 2251 (1962).

[8]Appendix 1 of reference 7.

[9]A more rigorous definition of $\langle x^{\alpha(t)}\rangle$ is given by Eq. (11) of reference 7.

[10]If high-energy scattering is described in terms of a purely imaginary potential which accounts for the dominant absorption at high energies, the phase of the amplitudes near the forward direction is strictly $\pm\frac{1}{2}\pi$, independent of the details of the imaginary potential. Therefore, the condition (3) depends primarily on the relative rates of change with energy of the real and imaginary parts of the effective potential.

[11]The estimate below gives only the order of magnitude. This, however, is quite sufficient. See also M. Froissart, Phys. Rev. 123, 1053 (1961). According to our notation, the upper limit due to Froissart is $\langle s^0\rangle$. We have shown in this note that only $\langle s^0\rangle$ and $\langle s^{-2}\rangle$ are consistent with analyticity, unitarity, and the purely imaginary forward amplitude at infinite energy.

[12]The latest experimental figure is $a_0+2a_3=-0.008\pm0.007$ [W. S. Woolcock, in Proceedings of the Aix-en-Provence International Conference on Elementary Particles, 1961 (C.E.N. Saclay, France, 1961), Vol. I, p. 459]. It is interesting to see how close this figure is to the borderline between two cases of different zeros.

MOVING POLES AND ELEMENTARY PARTICLES*

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An important problem in relativistic quantum field theory is the question whether all particles are to be described by moving poles (or Regge poles)[1] in the complex angular momentum plane, or whether the amplitudes can have single-particle poles and resonance poles in the energy variable which are not associated with angular momentum trajectories at all. In the present note, we use the general notions of relativistic dispersion theory and an assumption concerning the absence of a natural boundary in the complex angular momentum plane, in order to show that the interpolating partial-wave functions $F(s,\lambda)$ have no s-independent poles in the λ plane, at least for $\mathrm{Re}\lambda>-\frac{1}{2}$. This implies, for instance, that a constant total cross section for $t\to\infty$ in an elastic crossed channel must be due to a moving pole $\lambda=\alpha(s)$, $\alpha(0)=1$, because we cannot have $A_t(s,t)\sim C(s)t^{\alpha_0}$ with $\alpha_0=1$ in a whole neighborhood of $s=0$. Furthermore, we indicate, with some additional assumptions, that the invariant scattering amplitude $F(s,t)$ has no single-particle poles describing particles with spin larger than one which are not manifestations of a pole trajectory in the λ plane.

Let us consider the elastic scattering of two spin-zero particles with equal masses μ. We omit single-particle states with the same mass, and we assume that both particles are distinguishable. If the invariant amplitude $F(s,t)$ is analytic in the cut t plane and bounded by a polynomial of degree N for $t\to\infty$, then we can define functions $F_\pm(s,\lambda)$ by[2]

$$F_\pm(s,\lambda)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}dv\,\frac{1}{2q^2}Q_\lambda\left(1+\frac{v}{2q^2}\right)A_\pm(s,v),\quad(1)$$

where

$$A_\pm(s,v)=A_t(s,v)\pm A_u(s,v),\quad(2)$$

$4q^2=s-4\mu^2$, and A_t and A_u are the absorptive parts of $F(s,t)$ in the t and u channels, respectively. The real analytic function $F_\pm(s,\lambda)$ is regular for $\mathrm{Re}\lambda>N$, and for $\lambda=l>N$ (l = integer) it represents a unique interpolation of the physical partial wave amplitudes,

$$F_l(s)=\tfrac{1}{2}\int_{1}^{+1}dc\,F(s,t=-2q^2(1-c))P_l(c);\quad(3)$$

we have

$$F_\pm(s,l)=F_l(s)\quad\text{for } l=\text{even/odd},\ l>N.$$

In an earlier paper[3] we have shown that the possibility of continuing the function $F_\pm(s,\lambda)$ into the region $\mathrm{Re}\lambda<N$ depends entirely upon the properties of the absorptive part $A_\pm(s,v)$ for $v\to\infty$. For example, a term of the form

$$v^{\alpha(s)}(\ln v)^{\beta(s)}$$

in the asymptotic expansion of $A_\pm$ gives rise to a singular term in the λ plane which is of the form

$$F(s,\lambda)=\Gamma(s)/[\alpha(s)-\lambda]^{\beta(s)+1}+\dots\quad(4)$$

for $\beta \not\equiv -1$ and $F(s,\lambda) = \Gamma(s)\ln[\alpha(s)-\lambda] + \cdots$ for $\beta(s) \equiv -1$. In principle it could happen that the continuation of the function $F_\pm(s,\lambda)$ into the region $\mathrm{Re}\lambda < N$ is somewhere blocked by a natural boundary. We consider it improbable that such a curve of singularities exists in the λ plane, at least as far as the region $\mathrm{Re}\lambda > -\frac{1}{2}$ is concerned, and we shall assume in the following that the function $F_\pm(s,\lambda)$ is regular in the region $-\frac{1}{2} < \mathrm{Re}\lambda \leq N$ except for isolated singularities.[4]

Before we discuss the singularities in the λ plane, it is important to consider the function $F_\pm(s,\lambda)$ as an analytic function in s and λ. For $\mathrm{Re}\lambda > N$, the representation (1) is valid and, except for the branch points of $Q_\lambda[1+v/2q^2(s)]$, the region of analyticity of $F_\pm(s,\lambda)$ in the s plane is determined by that of the absorptive part $A_\pm(s,v)$ for $v \geq 4\mu$.[5]

For reasons of simplicity we assume that $A_\pm$ is regular in s except for the usual cuts and poles on the real axis.[6] In the region $-\frac{1}{2} < \mathrm{Re}\lambda < N$ we do not have the representation (1), and hence other singularities may appear in the s plane. However, as has been shown in reference 3, there cannot be any λ-independent singularities except those which are also present for $\mathrm{Re}\lambda > N$. As far as the λ-dependent singularities are concerned, it is very important to find out how poles and branch points disappear from the physical sheet of the s plane as we continue $F_+(s,\lambda)$ into the region $\mathrm{Re}\lambda > N$. This will be discussed in a separate note.

Let us now consider the possible isolated singularities of $F_\pm(s,\lambda)$ in the λ plane for $\mathrm{Re}\lambda > -\frac{1}{2}$. We want to show that there cannot be any s-independent poles, but only singular surfaces $\lambda = \alpha(s)$ with $\mathrm{Im}\alpha(s+i0) \neq 0$ for real s on the elastic cut $4\mu^2 < s < s_i$ (e.g., $s_i = 9\mu^2$). For this purpose we write the continued unitarity condition in the form

$$F_\pm(s+i0,\lambda) - F_\pm(s-i0,\lambda) = 2i\rho(s+i0)F_\pm(s+i0,\lambda) \times F_\pm(s-i0,\lambda), \qquad (5)$$

where

$$\rho(s) = (s-4\mu^2/s)^{1/2}, \quad \text{and } F_\pm^*(s^*,\lambda^*) = F_\pm(s,\lambda).$$

As in the case of integer angular momenta, we can continue $F_\pm(s,\lambda)$ through the elastic cut into a second sheet,[7] where we have

$$F_\pm^{\mathrm{II}}(s,\lambda) = F_\pm(s,\lambda)/[1+2i\rho(s)F_\pm(s,\lambda)], \qquad (6)$$

such that $F_\pm^{\mathrm{II}}(s \pm i0,\lambda) = F_\pm(s \mp i0,\lambda)$ for s real, and $4\mu^2 < s < s_i$.[8] Suppose now that $F_\pm(s,\lambda)$ has an s-independent pole for $\lambda = \lambda_0$, then we see from Eq. (6) that $F_\pm^{\mathrm{II}}(s,\lambda)$ has no such pole. The pole would have to disappear suddenly as we continue in s through the elastic cut from sheet I into sheet II. It follows from the continuity theorem[9,10] for functions of two or more complex variables that such a behavior of the analytic functions $F_\pm(s,\lambda)$ is not possible. Hence we have no s-independent poles for $\mathrm{Re}\lambda > -\frac{1}{2}$. For a pole trajectory $\lambda = \alpha_\pm(s)$, it follows from the unitarity condition that $\mathrm{Im}\alpha_\pm(s \pm i0) \neq 0$ for $4\mu^2 < s < s_i$. The real analytic function $\alpha_\pm(s)$ has a cut for $s \geq 4\mu^2$, and we have

$$F_\pm^{-1}(s,\alpha_\pm(s)) = F_\pm^{\mathrm{II}^{-1}}(s,\alpha_\pm^{\mathrm{II}}(s)) = 0.$$

We note that s-independent branch points are not excluded by the argument given above. In potential scattering, such branch points are known to occur if the potential contains a term proportional to r^{-2} which modifies the centrifugal term.[11] They are related to the "collapse into the center," and therefore one may expect that they are not present in the field theoretic case. If this is so, our argument shows that the high-energy behavior of the absorptive parts of $F(s,t)$ in the crossed channels (t and u channel) must be determined by a pole trajectory (or possibly a branch point trajectory, or even another moving singularity); the essential point being that we cannot have a constant power law like $A_t(s,t) \sim C(s)t^l$. Especially we see that a constant total cross section for $t \to \infty$ must be due to a moving pole, $\lambda = \alpha_0(s)$, with $\alpha_0(0) = 1$, because we cannot have $A_t(s,t) \sim tC(s)$ for a neighborhood of $s = 0$.

What are the implications of this result for the amplitude $F(s,t)$ itself? We assume that $F_\pm(s,\lambda)$ is sufficiently bounded for $|\lambda| \to \infty$, $\mathrm{Re}\lambda > -\frac{1}{2}$, so that we can write a Sommerfeld-Watson representation of the form

$$F_\pm(s,t) = -\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\lambda\,(2\lambda+1)\frac{\pi}{\sin\pi\lambda}F_\pm(s,\lambda) \times \tfrac{1}{2}[P_\lambda(-c) \pm P_\lambda(c)] + \sum_{l=0}^{N}(2l+1) \times [F_l(s) - F_\pm(s,l)]\tfrac{1}{2}[P_l(c) \pm P_l(-c)]$$
$$+ \sum_{\text{poles}} \frac{2\alpha_\pm(s)+1}{\sin\pi\alpha_\pm(s)}\beta_\pm(s)\tfrac{1}{2}\{P_{\alpha_\pm(s)}(c)\}, \qquad (7)$$

plus contributions from branch cuts and other moving singularities in the λ plane. Here $c=1+l/2q^2$, and $F_l(s)$ is the physical partial wave amplitude as defined by Eq. (3). Although $F_\pm(s,l)$ is uniquely determined by the partial waves for $l>N$, it is not necessarily equal to $F_l(s)$ for $l<N$. We note that along the elastic cut $4\mu^2<s<s_i$, both functions satisfy the unitarity conditions (5), and this gives rise to certain restrictions, but it does not exclude terms of the form

$$F_l(s)-F_\pm(s,l)\sim b(s)/(m^2-s), \tag{8}$$

where $b(s)$ is an analytic function with a cut along the real axis for $s\geq 4\mu^2$.

We suppose that the representation (7) is valid for s on and near the real axis. For $s\leq 0$ we are in the physical region of the t channel, and the bound[12] $|F_\pm(s,t)|<\text{const}\,t(\ln t)^2$ for $t\to\infty$ is applicable. Since it is impossible that there are cancellations between the different terms in Eq. (6), we conclude that $F_l(s)=F_\pm(s,l)$, $l=\text{even/odd}$, should be valid for $l>1$. Hence, if one accepts our assumptions, it appears that all single-particle poles of $F(s,t)$ which describe particles (or resonances) with spin larger than one must be due to moving poles in the λ plane. For $l\leq 1$ we cannot exclude the existence of poles like (8) which are not related to any trajectory. It would be very interesting to see whether a more exhaustive use of the unitarity condition could restrict or even eliminate the difference between $F_l(s)$ and $F_\pm(s,l)$ for $l\leq 1$.

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[1]T. Regge, Nuovo cimento 14, 951 (1959).

[2]M. Froissart, La Jolla Conference on the Theory of Weak and Strong Interactions, 1961 (unpublished); K. Bardakci, Phys. Rev. 127, 1832 (1962).

[3]R. Oehme and G. Tiktopoulos, Physics Letters 2, 86 (1962); this paper contains further references.

[4]For potential scattering with superpositions of Yukawa potentials, we know that $F_\pm(s,\lambda)$ has no natural boundary in the λ plane. Hence our theorem concerning the absence of s-independent poles shows that the poles of the Born approximation, $F_B(s,\lambda)\sim Q_\lambda[1+m^2/2q^2(s)]$, for negative integer values of λ, must start moving as higher orders are included. The situation in field theory may be similar, but there are some complications with the extension of our assumptions to the whole λ plane which we hope to discuss on another occasion.

[5]Note that in reference 3 we have included a factor $q_{(s)}{}^{-2\lambda}$ in the definition of $F_\pm(s,\lambda)$.

[6]S. Mandelstam, Phys. Rev. 112, 1344 (1958).

[7]R. Oehme, Phys. Rev. 121, 1840 (1961).

[8]Note that for noninteger values of λ, the function $F_\pm(s,\lambda)$ also has a branch cut for $s\leq 4\mu^2$.

[9]See, for instance, H. Behnke and P. Thullen, Theorie der Funktionen mehrerer complexer Veränderlichen (Springer-Verlag, Berlin, 1934).

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