the ratio $\sigma(E_1, t)/\sigma(E_2, t)$ at a moderate value of t].

(3) The connection which we have found between radiative effects and Regge poles puts the calculation of electromagnetic radiative corrections in a new perspective. For example, it is our knowledge of the fine structure of the positronium spectrum which leads us to believe that there probably are spin-dependent terms which should be summed in an exponential factor in the electron-proton scattering cross section. Also, the main part of the radiative corrections in a given process at high energy can probably be estimated directly from the known energy spectrum of the intermediate states allowed in the crossed channel.

(4) Finally, we would like to make a remark on the question of the origin of the Regge poles which has been widely discussed recently. In the theory of strong interactions, where we do not have a consistent field theory to start with, the question of whether Regge poles are "fundamental" or whether they can be derived by calculations based on a specific Lagrangian is rather a matter of taste (or, as has been said, a "philosophical" question). In quantum electrodynamics, the situation is different, because there exists a field theory enabling us to make predictions which can be tested experimentally. And since a Regge behavior of the cross sections can be obtained by a consistent high-energy approximation to the field equations, we do not see any reason at present to question the elementary nature of the photon.

We would like to thank Professor Hofstadter, Professor Tsai, and Professor Walecka for interesting discussions, and the Physics Department of Stanford University for its kind hospitality.

---

*Work supported in part by the U. S. Air Force Office of Scientific Research.

†Permanent address: Laboratoire de Physique Theorique et Hautes Energies, Boite Postale 12, Orsay, Seine-et-Oise, France.

[1]R. Blankenbecler, L. F. Cook, and M. L. Goldberger, Phys. Rev. Letters 8, 463 (1962).

[2]D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. 13, 379 (1961).

[3]J. Schwinger, Phys. Rev. 76, 790 (1949).

[4]S. Frautschi, M. Gell-Mann, and F. Zachariasen (to be published).

[5]H. A. Bethe and E. E. Salpeter, Handbuch der Physik (Springer-Verlag, Berlin, 1957), Vol. 35, p. 203. We conjecture that Eq. (7), with $\gamma + \nu = n$, gives to all orders the energy levels of a system of two spinless bosons with equal masses and opposite charges interacting relativistically through the electromagnetic field.

[6]L. Michel, Nuovo cimento 10, 319 (1953).

[7]A slope of the order of α^2/m^2 would still, however, be about 200 times larger than that of the vacuum trajectory.

[8]In addition to the exponential Regge behavior, it has been noted by Yennie et al. that the function $F(t)$ in Eq. (2) constitutes also a strong reduction factor for the high-energy cross section.

[9]F. Bloch and A. Nordsieck, Phys. Rev. 52, 54 (1937). It is apparent from Eq. (1), that making $\epsilon \to 0$ is equivalent to increasing $E \to \infty$.

REGGE POLES AND LANDAU SINGULARITIES

V. N. Gribov
A. F. Ioffe Physico-Technical Institute, Leningrad, U.S.S.R.

and

I. Ya. Pomeranchuk
Institute of Theoretical and Experimental Physics, Moscow, U.S.S.R.

(Received July 30, 1962)

A scattering amplitude[1] as a function of energy t and momentum transfer s is known to have singularities at threshold values of these variables and on Landau curves.[2] It is a remarkable property of Landau singularities that the threshold of one of the variables, e.g., t, is a line in the (s, t) plane which, as s increases to infinity, is asymptotic to an infinite number of Landau curves.[3] On the other hand, it has become clear recently that the asymptotic behavior of the scattering amplitude at high s is determined by analytic properties of partial-wave amplitudes $f_l(t)$, as functions of angular momentum l, in the channel where $\sqrt{t}$ is the total energy.[4] The problem is how the accumulation of Landau curves near the threshold value of t is manifested in the behavior of singularities of the amplitudes as functions of angular momentum l. It will be shown that at the energy $t = t_i$ corresponding to any two-particle threshold, the accumulation of an infinite number of poles on the line $\mathrm{Re}\,l = -\frac{1}{2}$ takes place. We shall present also some arguments for the statement that at the energy corresponding to an n-particle production threshold, the accumulation of an infinite number of poles must occur on the line $\mathrm{Re}\,l = -\frac{1}{2} - \frac{3}{2}(n-2)$.

This result means, in particular, that the invariant scattering amplitude for t equal to any n-particle threshold cannot decrease with increasing s faster than $1/\sqrt{s}$. This limitation on the rate of decrease of an amplitude is the consequence of unitarity and analyticity only. The proof that the accumulation of poles occurs uses the finiteness of the interaction range at fixed energy.

In spite of the accumulation of poles, one manages to calculate the asymptotic behavior of the absorptive part of the amplitude for t near $4\mu^2$ (μ being the pion mass). The contribution of the poles that accumulate on the line $\mathrm{Re}\,l = -\frac{1}{2}$ turns out to oscillate even for $t < 4\mu^2$. An oscillatory behavior for $t < 4\mu^2$ of the absorptive part of the amplitude describing the elastic scattering amplitude in the s channel is in contradiction with unitarity in the s channel,[5] because the unitarity condition in the s channel requires the positiveness of the absorptive part in the interval $0 < t < 4\mu^2$. Hence it follows necessarily that a partial-wave amplitude in the t channel in these cases must have at least one pole on the real axis to the right of $l = -\frac{1}{2}$ for small positive $4\mu^2 - t$. This result may be considered as a purely theoretical argument in favor of the necessity of the vacuum-pole existence. From the point of view of nonrelativistic quantum mechanics, the presence of a pole for $\mathrm{Re}\,l > -\frac{1}{2}$ means that the potential cannot be repulsive everywhere[6]: $\int urdr\varphi^2 > 0$. The statement made above means that an interaction compatible with unitarity and analyticity must be attractive in this sense.

Let us consider a partial-wave amplitude $f_l(t)$ for the scattering of identical spinless particles. For l on the half-plane to the right of all singularities of f_l in the complex l plane, the following relation holds[7]:

$$f_l(t) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} Q_l\left(1 + \frac{2s}{t-4\mu^2}\right) A_1(s,t)\,\frac{2s}{t-4\mu^2}, \qquad (1)$$

where $A_1(s,t)$ is the absorptive part of the amplitude in the s channel. $Q_l(x)$ is the Legendre function of the second kind. Hence, as $t \to 4\mu^2$,

$$f_l(t) = \frac{\Gamma(l+1)}{\pi^{1/2}2^{2l}\Gamma(l+\frac{3}{2})}(t-4\mu^2)^l \int_{4\mu^2}^{\infty} \frac{A_1(s,t)}{s^{l+1}}ds,$$

$$l \neq -\tfrac{1}{2}(2n+3), \quad n = 0, 1, \cdots. \qquad (2)$$

The partial wave $f_l(t)$ for $4\mu^2 < t < 16\mu^2$ satisfies the unitarity condition

$$(1/2i)[f_l(t) - f_{l^*}{}^*(t)] = (k/\omega) f_l(t) f_{l^*}{}^*(t),$$

$$k = \tfrac{1}{2}(t-4\mu^2)^{1/2}; \quad \omega = \tfrac{1}{2}\sqrt{t}. \qquad (3)$$

It follows from this relation that $\varphi_l(t) = (k/\omega)f_l(t)$ is less than unity for real l. On the other hand, according to (2), $\varphi_l(t)$ is proportional to $(t-4\mu^2)^{l+\frac{1}{2}}$.

If the Formula (2) is valid for $\mathrm{Re}\,l \leq -\frac{1}{2}$, then $\varphi_l(t)$ may become arbitrarily large for $t - 4\mu^2$ sufficiently small. This means that the Integral (2) must not exist for $\mathrm{Re}\,l \leq -\frac{1}{2}$, i.e., $A_1(s,t)$ at $t = 4\mu^2$ cannot decrease faster than $1/\sqrt{s}$, as s increases. As a consequence, $\varphi_l(t)$ must have singularities as $t \to 4\mu^2$ for $\mathrm{Re}\,l \geq -\frac{1}{2}$. Let us suppose that for $\mathrm{Re}\,l \geq -\frac{1}{2}$ there are only a finite number of poles at points λ_n. Then $\varphi_l(t)$ may be represented in the form

$$\varphi_l(t) = \tilde{\varphi}_l(t) = \textstyle\sum_n [r_n(t)/(l-\lambda_n)]\exp[-\delta(l-\lambda_n)]. \qquad (4)$$

Here $r_n(t)$ are the residues at the poles λ_n [$r_n \sim (t-4\mu^2)^{\lambda_n+\frac{1}{2}}$].[5] δ is an arbitrary positive number, and $\tilde{\varphi}_l(t)$ has no singularity for $\mathrm{Re}\,l \geq -\frac{1}{2}$. Owing to the presence of the exponential factor $\exp[-\delta(l-\lambda_n)]$ in (4), $\tilde{\varphi}_l$ decreases exponentially as $\mathrm{Re}\,l \to \infty$. $\tilde{\varphi}_l(t)$ may be represented in the form (2), where $A_1 = A_1 - \Delta A_1$ must be substituted for A_1, ΔA_1 being the contribution from the poles considered. The sum $\sum r_n \exp[-\delta(l-\lambda_n)]/(l-\lambda_n)$ being finite at $t \to 4\mu^2$, $\tilde{\varphi}_l(t)$ is also finite for real l in virtue of the unitarity condition. Hence we arrive at the conclusion that $\tilde{A}_1$ also cannot decrease faster than $1/\sqrt{s}$, and consequently $\tilde{\varphi}_l(t)$ must have singularities at $t = 4\mu^2$ for $\mathrm{Re}\,l \geq -\frac{1}{2}$, contrary to its definition. This means that $\varphi_l(t)$ cannot have a finite number of poles for $\mathrm{Re}\,l \geq -\frac{1}{2}$ and $t \to 4\mu^2$. It is easy to show that this conclusion would not be changed if $\varphi_l(t)$ had, in addition, any finite number of branch points for $\mathrm{Re}\,l \geq -\frac{1}{2}$.

Thus we arrive at the conclusion that as $t \to 4\mu^2$, $\varphi_l(t)$ must have an infinite number of singularities near the line $\mathrm{Re}\,l = -\frac{1}{2}$.

It easily can be seen that the same situation occurs at any two-particle threshold. For this purpose it is sufficient to consider the amplitude for any reaction where the particles which give rise to such a threshold are present in the initial or final state, because all these amplitudes, related by the unitarity condition, have common poles. An invariant partial amplitude f_{ab} for the transition of two particles into two is proportional to $k_a^l k_b^l$; k_a and k_b being the initial and final relative momenta. The amplitude with bounded modulus analogous to φ_l, $\varphi_{ab} = k_a^{\frac{1}{2}} f_{ab} k_b^{\frac{1}{2}}$, goes like $(k_a k_b)^{l+\frac{1}{2}}$, and consequently at the corresponding threshold value t ($k_a \to 0$ or $k_b \to 0$), we encounter the same phenomenon. It is interesting to note in this connection that since (because of the saturation of nuclear forces) nuclei with arbitrarily large masses of the order mA can exist in theory if the electromagnetic interaction is switched off, two-particle thresholds are present for arbitrarily large t. This means that the invariant amplitude A_1 cannot decrease faster than $1/\sqrt{s}$ for arbitrarily large $t \simeq (2mA)^2$.

Let us consider the situation near a many-particle threshold. With this purpose in mind, consider a partial amplitude for the transition of two particles into n particles with angular momentum l occurring in the unitarity condition for the elastic amplitude. Such an amplitude near the production threshold is proportional to

$$\left[t - \left(\sum_{i=1}^{n} m_i\right)^2\right]^{l/2}.$$

The amplitude with bounded modulus which is analogous to φ_{ab} is equal to $k_a f_{ab} \Gamma_b^{1/2}$, Γ_b being the phase-space volume for n particles, proportional to

$$\left[t - \left(\sum_{i=1}^{n} m_i\right)^2\right]^{\frac{1}{2}(3n-5)}.$$

Hence this limited amplitude is proportional to

$$\left[t - \left(\sum_{i=1}^{n} m_i\right)^2\right]^{\frac{1}{2}[l + \frac{1}{2}(3n-5)]}.$$

It is our opinion that such a dependence would give rise to the appearance of an infinite number of singularities near the line $\mathrm{Re}\,l = -\frac{1}{2}(3n-5)$ at $t \to (\sum_i m_i)^2$, and this fact would reflect the accumulation of Landau curves for $t = (\sum_i m_i)^2$ and $s \to \infty$. In connection with the preceding, the hypothesis of Predazzi and Regge[8] of the symmetry of the l plane with respect to the line $\mathrm{Re}\,l = -\frac{1}{2}$ (in which case the mentioned phenomenon does not occur) seems to us to be invalid if inelastic processes are taken into account, i.e., with this hypothesis one cannot see how the accumulation of Landau curves at a many-particle threshold are manifested in the l plane.

In order to explore in more detail the structure of singularities near the line $\mathrm{Re}\,l = -\frac{1}{2}$ as $t \to 4\mu^2$, we shall make use of the energy-independent boundary condition on the wave function which is valid in the nonrelativistic region $t \to 4\mu^2$ in the case of an exponentially decreasing interaction. This boundary condition, used usually for integer l, continues analytically into the complex l plane. If one represents the wave function outside the interaction region in the form

$$\psi_\nu = j_\nu(kr) + i\varphi_\nu(k) h_\nu^{(1)}(kr), \qquad (5)$$

where

$$j_\nu(x) = (\tfrac{1}{2}\pi x)^{1/2} J_\nu(x), \quad h_\nu^{(1)}(x) = (\tfrac{1}{2}\pi x)^{1/2} H_\nu^{(1)}(x),$$
$$\nu = l + \tfrac{1}{2}$$

[$J_\nu(x)$ and $H_\nu^{(1)}(x)$ are Bessel and Hankel functions], and designates $(r\psi_\nu'/\psi_\nu)|_{r=R}$ by χ_ν, one has, using the relation

$$h_\nu^{(1)} = (i/\sin\pi\nu)[\exp(-i\pi\nu) j_\nu - j_{-\nu}], \qquad (6)$$

that

$$\varphi_\nu = -\Lambda \sin\pi\nu/[1 - \exp(-i\pi\nu)\Lambda], \qquad (7)$$

where

$$\Lambda = [j_\nu'(kR) - \chi_\nu j_\nu(kR)]/[j_{-\nu}'(kR) - \chi_\nu j_{-\nu}(kR)]. \qquad (8)$$

As $k \to 0$ ($\nu \neq 0, \pm 1, \pm 2, \cdots$),

$$\Lambda = (kR)^{2\nu}[\chi_\nu - \nu\Gamma(1-\nu)]/[\chi_\nu + \nu\Gamma(1+\nu)], \qquad (9)$$

and χ_ν does not depend on the energy. Hence it follows that for low k, Λ oscillates rapidly along the line $\mathrm{Re}\,\nu = 0$ and its modulus changes rapidly when ν deviates from this line. Therefore, φ_ν must have an infinite number of poles.

We consider in more detail the region $\nu \ll 1$. If the coefficient of $(kR)^{2\nu}$ is expanded as a power series in ν, then the linear term may be eliminated by means of redefining R. Then

$$\Lambda = x^{2\nu}(1 + \gamma\nu^2), \qquad (10)$$

where $x = ka$ and a is the redefined interaction radius. To eliminate $\exp(-i\pi\nu)$ in the denominator of (7), we consider $t < 4\mu^2$. Then the equation for determining the position of the poles has the form

$$x^{-2\nu} = 1 + \gamma\nu^2, \quad x = a\kappa, \quad \kappa = \tfrac{1}{2}(4\mu^2 - t)^{1/2};$$

$$-\nu\tau = \gamma\nu^2 + 2i\pi n, \quad n = 0, \pm 1, \cdots, \quad \tau = \ln x^2; \tag{11}$$

$$\nu_n = (2i\pi n/\tau) + 4\pi^2 n^2 \gamma/\tau^3. \tag{12}$$

Thus we have obtained an infinite number of complex conjugate poles. As $\tau \to -\infty$, poles are situated to the left of the line $\mathrm{Re}\nu = 0$ for $\gamma > 0$. It is to be noted that in nonrelativistic quantum mechanics, $\gamma \geq 0$ for an arbitrary potential, because for $\mathrm{Re}\nu > 0$ and $t < 4\mu^2$, complex poles do not exist.[9] The case $\gamma = 0$ corresponds to the potential increasing at small distances faster than $1/r^2$.

We shall suppose in the following that $\gamma > 0$. Formulas (7) and (12) make it possible to calculate the asymptotic behavior of $A_1(s, t)$ for small $t - 4\mu^2$ and large s. $A_1(s, t)$ contains the contribution from a finite number of the poles, situated to the right of the line $\mathrm{Re}\nu = 0$. We are not interested in this contribution. Designating the contribution from the poles to the left of the line $\mathrm{Re}\nu = 0$ by $\tilde{A}(s, t)$, we obtain

$$\tilde{A}_1(s, t) = -\frac{1}{2i}\left(\frac{4\mu^2}{s}\right)^{1/2} \int_{-i\infty}^{i\infty} \frac{d\nu \exp(\xi\nu)\nu(1 + \gamma\nu^2)}{1 - \exp(\tau\nu)(1 + \gamma\nu^2)}, \tag{13}$$

where $\xi = \ln sa^2$.

We made use in (13) of the approximate expression for φ_ν applicable for $\nu \ll 1$. The reason is that, as will be seen in the following, only low ν are of importance in the integral (13) for large ξ.

After closing the integration contour on the left and computing the residues, we obtain, neglecting terms of order $1/\tau^2(\tau\xi)^{1/2}$ compared to unity,

$$\tilde{A}_1 = -4\pi^2 \left(\frac{4\mu^2}{s}\right)^{1/2} \frac{1}{\tau^2} \sum_{n=1}^{\infty} \exp\left(\frac{4\pi^2 n^2 \gamma\xi}{\tau^3}\right) n \sin 2\pi n \frac{\xi}{\tau}. \tag{14}$$

We consider now the case $\xi \gg \tau^3$. Then only one term is of importance in the sum (14), and

$$\tilde{A}_1 = -4\pi^2 \left(\frac{4\mu^2}{s}\right)^{1/2} \frac{1}{\tau^2} \exp\left(\frac{4\pi^2 \gamma\xi}{\tau^3}\right) \sin 2\pi \frac{\xi}{\tau}. \tag{15}$$

Thus we have obtained an oscillating behavior for $\tilde{A}_1$. The consequences of this result have been discussed already in the beginning.

In the course of this discussion we have noted that the interaction corresponding to repulsion in nonrelativistic quantum mechanics contradicts analyticity and unitarity. At the same time an arbitrarily small attraction conforms to unitarity and analyticity, because, as can be shown easily, for arbitrarily small attraction, there is a pole on the real axis at $l > -\frac{1}{2}$, $t = 4\mu^2$.[6] As the interaction strength decreases, the position of the pole at $t = 4\mu^2$ tends to $l = -\frac{1}{2}$. It is of interest to note also that the complex-conjugate poles discussed above go to $-\infty$ in the l plane, when the interaction strength tends to zero. One can be convinced of this by calculating γ, which occurs in (10), by making use of perturbation theory which is valid for finding χ for weak potentials. Then one can show easily that $\gamma = 1/g^4$, where $g^2 = -\int urdr$. Hence it follows that the asymptotic expression for the scattering amplitude (15) has a power-series expansion in g^2 identically equal to zero.

Up to now we have discussed complex-conjugate poles for $\mathrm{Re}\nu < 0$. The question arises how Formula (7) contains the possibility of the presence of poles on the real axis, if $\Lambda = \alpha_\nu(4\mu^2 - t)^\nu$ is equal to zero for $\mathrm{Re}\nu > 0$ and to infinity for $\mathrm{Re}\nu < 0$. It is obvious that the position of such poles φ_ν for $t = 4\mu^2$ coincides with that of α_ν for $\nu > 0$ and with the zeros of α_ν for $\nu < 0$. Starting from these reasons one can easily obtain the formula for $\nu_n(t)$ near $t = 4\mu^2$. This being done for $\nu_n(4\mu^2) = \beta_n > 0$, we obtain a result coinciding with that obtained in (4). In the case of poles with $\nu < 0$, setting

$$\alpha_\nu = (1/\rho_n)(\nu - \beta_n)(1/\sin\beta_n),$$

and substituting it into (7), one can show easily that $\varphi_\nu(t)$ has a pole at

$$\nu_n(t) = \beta_n + \rho_n(4\mu^2 - t)^{-\beta_n}/\sin\pi\nu, \tag{16}$$

and that the residue at this pole is equal to ρ_n.

If ν is equal to an integer so that $\Lambda = (-1)^\nu$, then substituting Neumann functions for the j_ν and taking into account their behavior at small k, one obtains

$$\nu_n(t) = \beta_n(t) + \rho_n^{-1}(4\mu^2 - t)^{-\beta_n} \ln[(4\mu^2 - t)/4\mu^2]. \tag{17}$$

Comparing (16) and (17) with reference 5, we see that the way by which the pole moves away from the real axis is determined by the modulus of ν only. Note that according to (17) the pole can move away from the real axis at $\nu = 0$ only if $\alpha = 0$, but in this case the residue vanishes.

To conclude, we should like to thank Ya. Ya. Azimov, V. B. Berestetsky, V. M. Shekhter,

and I. M. Shmushkevich for useful discussions.

[1]We use this term, "scattering amplitude," for the sake of brevity. The whole following consideration is true for any invariant amplitude for the transition of two particles into two particles.

[2]L. D. Landau, Nuclear Phys. 13, 181 (1959).

[3]R. Karplus, C. M. Sommerfield, and E. H. Wichman, Phys. Rev. 111, 1187 (1958).

[4]T. Regge, Nuovo cimento 14, 951 (1959); 17, 947 (1960); V. N. Gribov, J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 667 (1961) [translation: Soviet Phys.-JETP 14, 478 (1962)]; 41, 1962 (1961) [translation: Soviet Phys.-JETP 14, 1395 (1962)]. G. F. Chew and S. C. Frautschi, Phys. Rev. Letters 5, 580 (1961); R. Blankenbecler and M. L. Goldberger, Phys. Rev. 126, 766 (1962); S. C. Frautschi, M. Gell-Mann, and F. Zachariasen (to be published); S. C. Frautschi, G. F. Chew, and S. Mandelstan (to be published).

[5]V. N. Gribov and I. Ya. Pomeranchuk, J. Exptl. Theoret. Phys. (U.S.S.R.) 43, 310 (1962).

[6]T. Regge, Nuovo cimento 17, 951 (1960).

[7]V. N. Gribov, J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 1962 (1961) [translation: Soviet Phys.-JETP 14, 1395 (1962)].

[8]E. Predazzi and T. Regge, Nuovo cimento 24, 518 (1962).

[9]T. Regge, Nuovo cimento 14, 951 (1959).

ERRATUM

DEPARTURES FROM ONE-PION EXCHANGE IN 1.25-BeV π^--p INTERACTIONS. E. Pickup, D. K. Robinson, and E. O. Salant [Phys. Rev. Letters 9, 170 (1962)].

Third paragraph, first sentence: The inequalities are reversed. The ratio f/b is correctly defined in Table I.

Fourth paragraph, beginning "Dependence of...": Second sentence should read: "Figure 2 shows, for events of low Δ^2 inside the ρ peak, the distribution of events for forward α and for backward α." Fourth sentence should read: "For events inside the ρ peak with $\Delta^2 < 0.30$ (BeV/c)^2, the scattering asymmetry parameter $(F-B)/(F+B)$ (where F = number of pions scattered in the range $0 \leq \cos\theta < 1$, B = number of pions scattered in the range $-1 \leq \cos\theta < 0$) is 0.40 ± 0.08 for forward α and 0.08 ± 0.09 for backward α."