MAGNETIC-FIELD DEPENDENCE OF THE VELOCITY OF SOUND IN METALS

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Bohm and Staver[1] have shown how to estimate the magnitude of the velocity of longitudinal acoustic waves in metals by considering the screening of the motion of the ions by the conduction electrons. In their work a metal is assumed to consist of the plasma formed by the positive ions and the conduction electrons interacting by means of Coulomb forces. It was found that a disturbance of wave vector $\vec{q}$ propagates with angular frequency ω such that

$$\omega^2 = \Omega_p^{\;2}/\epsilon(\vec{q}, \omega), \tag{1}$$

where Ω_p is the plasma frequency characteristic of the motion of the ions when the conduction electrons are assumed to be immobile, and $\epsilon(\vec{q}, \omega)$ is the dielectric constant[2] appropriate to the wave vector $\vec{q}$ and the frequency ω. From (1) and using the expression (see reference 2) for $\epsilon(\vec{q}, \omega)$ in the limit of long wavelength ($q \to 0$), one obtains the equation

$$s_0 = (zm/3M)^{1/2} v_0, \tag{2}$$

which relates the velocity of sound s_0 to the velocity v_0 of an electron on the surface of the Fermi sphere; z is the number of conduction electrons per atom, and m and M are the masses of the electron and the atom, respectively.

The purpose of this Letter is to investigate the change in the velocity of sound when the metal in question is in the presence of a dc magnetic field of induction $\vec{B}_0$. We shall use the model of reference 1 and concern ourselves with temperatures near absolute zero. In general, the dielectric constant is a tensor whose components are functions of B_0. However, for longitudinal waves propagating in the direction of $\vec{B}_0$ (i.e., $\vec{q}$ is parallel to $\vec{B}_0$; we choose a Cartesian system of coordinates whose z axis is parallel to $\vec{B}_0$), Eq. (1) is modified in the simplest fashion by replacing $\epsilon(\vec{q}, \omega)$ by $\epsilon_{zz}(\vec{q}, \omega)$. We show that, in this geometry, the velocity of sound is an oscillatory function of the magnetic field.

The calculation of $\epsilon_{zz}(\vec{q}, \omega)$ is carried out using the method of the self-consistent field,[3] in which the equation of motion of the one-electron density matrix[4] is solved to first order in the self-consistent potential. One finds the following expression for the zz component of the dielectric constant:

$$\epsilon_{zz}(\vec{q}, \omega) = 1 + \frac{m\omega_p^{\;2}}{Nq^2} \sum_{nk_yk_z} \{[E_n(k_z+q) - E_n(k_z) - \hbar\omega]^{-1} + [E_n(k_z+q) - E_n(k_z) + \hbar\omega]^{-1}\},$$
$$E_n(k_z) < E_0. \tag{3}$$

In this relation the symbols nk_yk_z designate the quantum numbers that characterize the stationary states[5] of an electron in the presence of the magnetic field B_0 and the quantities

$$E_n(k_z) = \hbar\omega_0(n+\tfrac{1}{2}) + \hbar^2 k_z^{\;2}/2m \tag{4}$$

are the corresponding energy eigenvalues. The quantum number n is a non-negative integer, ω_0 and ω_p are the cyclotron and plasma frequencies of the electrons, respectively, and N is the number of conduction electrons in the volume Ω of the sample. The sum in Eq. (3) extends over all states having energy below the Fermi energy E_0. The allowed values of k_y and k_z are determined by periodic boundary conditions.

Expression (3) can be easily evaluated in the long-wavelength limit ($q \to 0$). We obtain

$$\epsilon_{zz}(\vec{q}, \omega) = 1 + \frac{3\omega_p^{\;2}\omega_0}{2q^2 v_0^{\;3}} \sum_n \left(\frac{1}{K_n - m\omega/\hbar q} + \frac{1}{K_n + m\omega/\hbar q} \right). \tag{5}$$

Here

$$K_n = (2m/\hbar^2)^{1/2}[E_0 - (n+\tfrac{1}{2})\hbar\omega_0]^{1/2}, \tag{6}$$

and the sum over n extends from $n=0$ to $n=n_0$, the largest integer for which K_n is real. The velocity of sound s thus satisfies the implicit equation

$$s^2 = \tfrac{2}{3} \left(\frac{\Omega_p}{\omega_p} \right)^2 \frac{v_0^{\;3}}{\omega_0} \left[\sum_n \left(\frac{1}{K_n - ms/\hbar} + \frac{1}{K_n + ms/\hbar} \right) \right]^{-1}. \tag{7}$$

Several regions of interest may be considered depending on the magnitude of the dimensionless parameters $a = (mv_0^{\;2}/2\hbar\omega_0)$ and $b = (ms_0^{\;2}/2\hbar\omega_0)$.

(a/b is independent of the magnetic field[6] and is of the order of 10^5.)

In the quantum limit, i.e., when $a < (\frac{3}{2})^{2/3}$, all the electrons are in the first Landau level ($n = 0$), and we have

$$s = \tfrac{1}{3}[zm/(zm + M)]^{1/2}(mv_0^3/\hbar\omega_0). \quad (8)$$

For ordinary metals this formula becomes applicable for magnetic fields larger than 10^8 gauss. However, the region in which b is of the order of unity or larger is well within the range of attainable magnetic fields. For very weak magnetic fields ($b \gg 1$) we find

$$s = s_0[1 - \tfrac{1}{8}(ab^2)^{-1/2}g(E_0/\hbar\omega_0)]. \quad (9)$$

The function $g(E_0/\hbar\omega_0)$ is periodic in the argument with period equal to unity. It is convenient to express $E_0/\hbar\omega_0$ in the form

$$E_0/\hbar\omega_0 = n_0 + \tfrac{1}{2} + \Delta, \quad (10)$$

where Δ lies between 0 and 1. Therefore g depends only on Δ. In general, we need not distinguish between the Fermi energy E_0 in the presence of the magnetic field and the field-free Fermi energy ζ_0 (except in the quantum limit), because their fractional difference is of the order of $a^{-3/2}$ which is negligible as compared to the order of magnitude of the quantities that we retain, namely, $a^{-1/2}$ and $(ab^2)^{-1/2}$. Because of the properties of the function g, the velocity of sound given in (9) exhibits an oscillatory variation with the magnetic field with period proportional to B_0^{-1}. The physical origin of this effect is the same that gives rise to the de Haas–van Alphen oscillations of the magnetic susceptibility of metals. For a typical metal and $B_0 = 1000$ gauss, $a \simeq 4 \times 10^5$ while $b \simeq 5$, so that the amplitude of the oscillations in the velocity of sound is of the order of $\frac{1}{8}(ab^2)^{-1/2} \simeq \frac{1}{2} \times 10^{-4}$ which is possible to detect. However, the period of the oscillation is so small that for good resolution it is required that the magnetic field in the sample be uniform within at least two parts per million. This would probably make this effect extremely difficult to observe except perhaps in the case of semimetals where the period is longer. The results of the calculation of the function g are displayed in Fig. 1. Because for higher magnetic fields the resolution is considerably increased, a numerical solution of Eq. (7) for $b = 0.1$ and $a = 10^4$ was carried out and plotted in Fig. 2. In these calculations use is made of the second Eulerian sum formula[7] retaining the first three terms.

Finally, it is worth remarking that in practice this effect can be observed probably only at liquid-helium temperatures and for rather pure samples, because unless the electron relaxation time $\tau \gtrsim \omega_0^{-1}$, the resulting broadening of the Landau levels smears out the oscillations.

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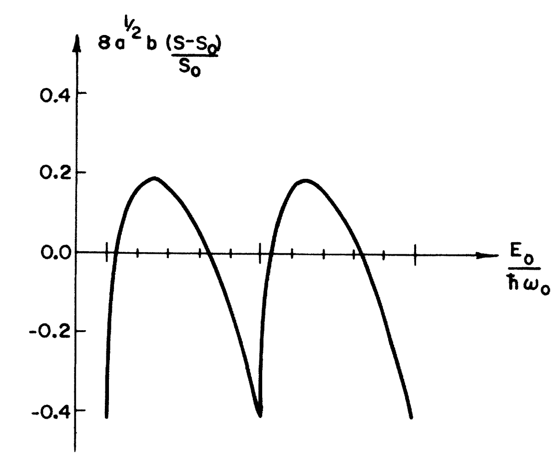

FIG. 1. Fractional change in the velocity of sound as a function of the magnetic field in the region in which $b \gg 1$. The abscissas are indicated starting from an arbitrary half-integral value of $E_0/\hbar\omega_0$ in the appropriate range (of the order of 4×10^5) and increasing in steps of unity. This graph is plotted using $g(\Delta) \approx \frac{4}{3}(\frac{1}{2} + \Delta)^{3/2} - \frac{1}{24}(\frac{1}{2} + \Delta)^{-1/2} + \frac{7}{7680}(\frac{1}{2} + \Delta)^{-5/2} - 2\Delta^{1/2}$, which is obtained by using the second Eulerian sum formula[7] and retaining only the first three terms.

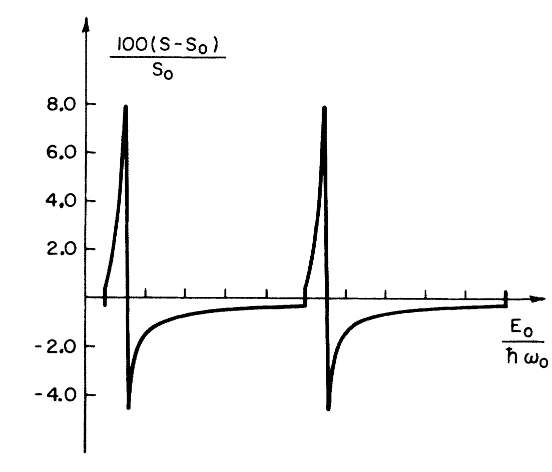

FIG. 2. Fractional change in the velocity of sound as a function of magnetic field for $b = 0.1$ and $a = 10^4$. The abscissas are again indicated in steps of unity starting from the half-integral value of a nearest to 10^4.

locity of sound in a transverse magnetic field.

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[4]See, for example, C. Kittel, Elementary Statistical Physics (John Wiley & Sons, Inc., New York, 1958), pp. 107-110.

[5]L. D. Landau, Z. Physik 64, 629 (1930).

[6]In the numerical estimates that follow, we assume the Fermi energy in the absence of a magnetic field $\zeta_0 = \frac{1}{2}mv_0^2 = 5$ eV and $s_0 = 5\times10^5$ cm/sec.

[7]F. B. Hildebrand, Introduction to Numerical Analysis (McGraw-Hill Book Company, Inc., New York, 1956), p. 154.

TUNNELING INTO SUPERCONDUCTORS*

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In a recent Letter, Cohen, Falicov, and Phillips[1] have discussed tunneling of electrons through a thin insulating layer between a normal and a superconducting metal on the basis of an effective Hamiltonian,

$$H = H_n + H_S + H_T, \tag{1}$$

where H_n and H_S are exact Hamiltonians for the normal and superconducting metals, respectively, and H_T is an operator which transfers electrons from one to the other:

$$H_T = \sum_{k,q,\sigma} (T_{kq} c_{k\sigma}^* c_{q\sigma} + T_{kq}^* c_{q\sigma}^* c_{k\sigma}). \tag{2}$$

Here k is a quantum number describing states in the normal metal, q refers to states in the superconductor, σ is the spin, and the c's are creation and destruction operators for normal quasi-particle states in both metals. By making use of the equations of motion, they derived an expression for the time rate of change of number of electrons in the superconductor $\langle N_S\rangle$ and thus the tunneling current. They find that the ratio of tunneling currents in superconducting and normal states depends only on the density of states in energy in the superconductor, as indicated by the experiments.[2]

We would like to discuss their derivation from a somewhat different point of view, which we feel brings out a little more clearly the connection with the semiconductor model of a superconductor and also with an earlier discussion of tunneling by the present author.[3] In the semiconductor model, one assumes that there is a set of normally occupied quasi-particle states below the gap and a set of normally unoccupied states above the gap, in one-to-one correspondence with those of the normal metal. At $T = 0°$K, states above are all unoccupied, those below occupied, but at a finite temperature electrons may be thermally excited to states above the gap, leaving holes in the normally occupied band. Electrons may be transferred from the normal to the superconducting metal into unoccupied states above the gap or into holes below the gap. Correspondingly, transfer in the reverse direction occurs from occupied states above the gap or from one of the filled states below the gap, leaving holes behind. It is the occupied states above the gap and the holes below which correspond to quasi-particle excitations of the superconductor.

What the author showed in his earlier Letter is that if there is a one-to-one correspondence between the quasi-particle excitations in normal and superconducting states, the only significant factor in the tunneling current is given by the density of states in energy. However, justification for the one-to-one correspondence and the definition of the quasi-particle states from microscopic theory was not given.

The quasi-particle states in a superconductor are usually defined by the Bogoliubov-Valatin transformation,[4]

$$\gamma_{q\uparrow}^* = u_q c_{q\uparrow}^* - v_q c_{-q\downarrow}, \tag{3a}$$

$$\gamma_{-q\downarrow}^* = u_q c_{-q\downarrow}^* + v_q c_{q\uparrow}, \tag{3b}$$

where $u_q^2 = 1 - v_q^2 = \frac{1}{2}(1 + \epsilon_q/E_q)$; $E_q = (\epsilon_q^2 + \Delta^2)^{1/2}$, and $-q$ indicates the time-reversal conjugate of q. These operators do not conserve particle number and are designed to operate on wave