

## Fidelity Decay as an Efficient Indicator of Quantum Chaos

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We demonstrate that a system's rate of fidelity decay under repeated perturbations may be measured efficiently on a quantum information processor, and analyze the conditions under which this indicator is a reliable probe of quantum chaos. The type and rate of the decay are not dependent on the eigenvalue statistics of the unperturbed system, but depend on the system's eigenvector statistics in the eigenbasis of the perturbation. For random eigenvector statistics, the decay is exponential with a rate fixed by the variance of the perturbation's energy spectrum. Hence, even classically regular models can exhibit an exponential fidelity decay under generic quantum perturbations. These results clarify which perturbations can distinguish classically regular and chaotic quantum systems.

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Over the past two decades, a great deal of insight has been achieved regarding the manifestations of chaos and complexity in quantum systems. We are interested in the problem of identifying such signatures in the context of quantum simulation on a quantum information processor (QIP). It is known that a QIP enables efficient simulation of the dynamics of a wide class of quantum systems [1,2]. In the case of quantum chaos models, quantum simulation provides an exponential speedup over direct classical simulation [3–5]. Recently, the quantum baker's map has been implemented using a nuclear magnetic resonance QIP [6]. These developments highlight the importance of devising efficient QIP methods for the measurement of quantum chaos signatures and related properties of complex quantum systems.

Perhaps the most established signature of quantum chaos is given by the (nearly) universal correspondence between the eigenvalue [7] and eigenvector [8] statistics of quantized classically chaotic systems and those of the canonical ensembles of random matrix theory (RMT). Unfortunately, direct detection of these spectral signatures is algorithmically inefficient by any known technique. However, following the original observation of Peres [9], some recent work has demonstrated that, under sufficiently strong perturbation, the fidelity exhibits a characteristic exponential decay in the case of classically chaotic quantum systems [10–13]. Below, we demonstrate that this characteristic decay may be measured by an efficient algorithm, and analyze in detail how the observation of this decay may be applied as an indicator of canonical RMT statistics (quantum chaos) in the *unperturbed* system.

By first considering RMT models, we show that Wigner-Dyson fluctuations in the system eigenvalue spectrum are not necessary to produce this characteristic decay. More importantly, we find that the canonical RMT statistics for the system eigenvector components, expressed in the eigenbasis of the perturbation operator,

are sufficient to produce the characteristic exponential decay. These observations are checked in the case of a dynamical model, where we demonstrate that an exponential fidelity decay can arise not only for a classically chaotic system, but also for a classically regular system under a generic choice of perturbation. Our analysis implies specific restrictions that must be imposed on the choice of applied perturbation operator in order to extract useful information about the statistical signatures of quantum chaos in the unperturbed system.

The aim of this Letter is to characterize certain static properties of a unitary map,  $U$ , by observing the rate of divergence between a fiducial Hilbert space vector evolved under this map,  $|\psi_u(n)\rangle = U^n|\psi(0)\rangle$ , and the same initial vector evolved under this map but subject also to a sequence of small perturbations  $|\psi_p(n)\rangle = (U_p U)^n|\psi(0)\rangle$ , where  $U_p = \exp(-iV\delta)$  is some unspecified perturbation operator and  $n$  denotes the number of iterations. The fidelity,

$$O(n) = |\langle\psi_u(n)|\psi_p(n)\rangle|^2, \quad (1)$$

provides a natural indicator of this divergence. The value of  $O(n)$  may be determined by an efficient algorithm on a QIP as follows. We start by preparing the fiducial state  $|\psi(0)\rangle = U_o|0\rangle^{\otimes n_q}$ , where  $n_q$  is the number of qubits required to span the system's Hilbert space  $N = 2^{n_q}$ . As shown below, the choice of fiducial state is not critical and the computational basis states, which are simplest to implement, are a convenient set. After applying the sequence  $(U^\dagger)^n(U_p U)^n$ , the system register contains a final state  $|\psi(n)\rangle$ . The circuit implementation of  $U$  requires only Poly( $n_q$ ) operations for the simulation of a wide class of quantum systems to arbitrary accuracy [1–5]. Here we observe that  $|\langle\psi_p(n)|\psi_u(n)\rangle| = |\langle\psi(n)|\psi(0)\rangle| = |\langle\psi_f|0\rangle|$ , where the state  $|\psi_f\rangle$  is obtained by time reversing the initial state preparation  $|\psi_f\rangle = U_o^\dagger|\psi(n)\rangle$ . The magnitude of  $O(n)$  is then determined from sampling the

population of the state  $|0\rangle$ . The entire algorithm therefore scales as  $\text{Poly}(n_q)$ .

Recently, Jacquod and co-workers [11] observed that, for  $|\psi(0)\rangle = |v_o\rangle$ , an initial eigenstate of  $U$  with eigenphase  $\phi_o$ , the fidelity relates to the local density of states (LDOS),  $\eta(\phi_o - \phi'_m) = |\langle v_o | v'_m \rangle|^2$ , via Fourier transform,

$$O(n) = \left| \sum_m \eta(\phi_o - \phi'_m) \exp[-i(\phi_o - \phi'_m)n] \right|^2. \quad (2)$$

In the above,  $|v'_m\rangle$  are eigenstates of the perturbed map  $U_p U |v'_m\rangle = \exp(-i\phi'_m) |v'_m\rangle$ . These observations locate the decay of  $O(n)$  in an existing theoretical framework. In the nonperturbative regime  $\sigma/\Delta \gtrsim 1$ , where  $\sigma^2 = \delta^2 V_{mn}^2$  denotes a typical off-diagonal matrix element and  $\Delta$  is the average level spacing, previous work suggests that when the perturbed system is complex the LDOS is typically Lorentzian [14,15]:

$$\eta(\phi_o - \phi'_m) \propto \frac{\Gamma}{(\phi_o - \phi'_m)^2 + (\Gamma/2)^2}, \quad (3)$$

with width  $\Gamma = 2\pi\sigma^2/\Delta$  determined by the Fermi golden rule (FGR). From (2) and (3), one expects the exponential decay,

$$O(n) \simeq \exp(-\Gamma n). \quad (4)$$

The onset of the exponential decay (4) has been confirmed recently in a few classically chaotic model systems [11,12], though the rate is not always given by the golden rule [10,16]. However, in the case of integrable  $U$ , the situation is less clear since under some perturbations the LDOS is known to also take on a Lorentzian shape [15,17].

Below, we examine which statistical properties of the *unperturbed* system lead to the FGR decay and how this characteristic decay depends on the choice of perturbation operator. This approach is motivated by the context of QIP simulation, in which the eigenbasis of the perturbation may be mapped onto an arbitrary basis of the simulated system [18]. Our choice of perturbation eigenvalue structure is also motivated from the perspective of quantum control studies. Specifically, we consider

$$U_p = \prod_{j=1}^{n_q} \exp(-i\delta\sigma_z^j/2), \quad (5)$$

where  $\sigma_z$  is the usual Pauli matrix and (5) therefore corresponds to a collective rotation of all the qubits by an angle  $\delta$ . Equation (5) is a model of coherent far-field errors [19], and for this type of error model a better understanding of the fidelity decay is a subject of intrinsic interest.

As a first test of the LDOS/FGR framework in the case of the qubit perturbation (5), we evaluate the fidelity decay for a map  $U = U_{\text{CUE}}$  drawn from the circular unitary ensemble (CUE). These matrices form an established model for classically fully chaotic time-periodic

systems (without additional symmetries) since these systems (almost always) exhibit the same characteristic (Wigner-Dyson) spectral fluctuations and eigenvector statistics as the CUE [20]. The randomness of the system eigenvectors enables a system-independent estimate of the rate  $\Gamma$  of the FGR decay. Since the components of random eigenstates are distributed uniformly over the basis states and uncorrelated with the distribution of eigenvalues, the second moment of the matrix elements  $V_{mn}$  may be directly evaluated,

$$\overline{V_{mn}^2} = \overline{\lambda^2}/N, \quad (6)$$

assuming  $\overline{\lambda} = 0$  and where  $\overline{\lambda^2} = N^{-1} \sum_{i=1}^N \lambda_i^2$  denotes the variance of the eigenvalues of  $V$ . As a result, the rate of the FGR decay is determined by the eigenvalues of the perturbation,

$$\Gamma = \delta^2 \overline{\lambda^2}, \quad (7)$$

where we have used  $\Delta = 2\pi/N$ . For the qubit perturbation (5), the variance of the eigenvalues has a simple form,

$$\overline{\lambda^2} = \frac{1}{N} \sum_{k=0}^{n_q} \left( \frac{2k - n_q}{2} \right)^2 C_k^{n_q}, \quad (8)$$

where the  $C_k^{n_q}$  are binomial coefficients. Using our RMT estimate (7), for  $n_q = 10$ , the rate is

$$\Gamma = 2.50\delta^2. \quad (9)$$

While a CUE map may be generated on a QIP using the gate decomposition devised in Ref. [21], for our numerical study we construct  $U = U_{\text{CUE}}$  directly from the eigenvectors of a random Hermitian matrix. Since computational basis states are easiest to prepare in the QIP setting, we consider the fidelity decay for both single computational basis states and averages over 50 such states. The behavior of the fidelity decay for a matrix typical of CUE is displayed in Fig. 1. The three perturbation values displayed in the figure are chosen near the onset of the nonperturbative regime ( $\delta > 0.1$ ), and it is evident that the fidelity decay even for *individual* computational basis states exhibits FGR decay (4) at the expected rate. The FGR decay persists for a time scale  $\Gamma^{-1} \log(N)$  until saturation at a time average that decreases as  $1/N$ .

We next consider  $O(n)$  for the Gaussian unitary ensemble (GUE) in order to clarify the relationship between the FGR decay and the distinct statistical features of RMT that represent signatures of quantum chaos. The GUE consists of Hermitian matrices with independent elements drawn randomly with respect to the unique unitarily invariant measure [20]. GUE forms the relevant RMT model for the important class of chaotic or complex autonomous Hamiltonian systems that are unrestricted by any additional symmetries. We may examine the sensitivity to perturbations for the GUE by constructing the unitary operator  $U_{\text{GUE}} = \exp(-iH_{\text{GUE}}\tau)$ , where  $\tau$  is a

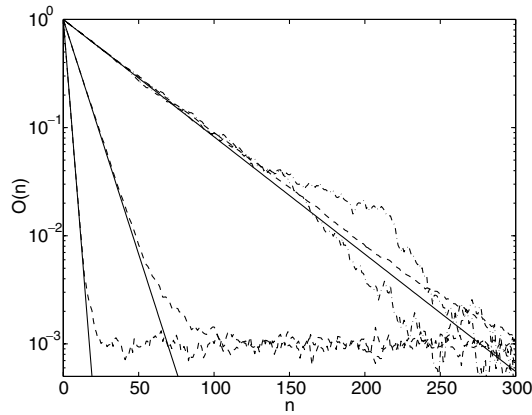


FIG. 1. Fidelity decay for  $U_{\text{CUE}}$  averaged over 50 computational basis states (dashed lines) is in excellent agreement with the golden rule decay (4) and the RMT rate (9) for  $\delta = (0.1, 0.2, 0.4)$  (solid lines top to bottom). Chain lines show the fidelity decay for two typical computational basis states in the case  $\delta = 0.1$ .

time delay between perturbations. We consider the same perturbation as for the CUE case. For sufficiently small  $\tau$ , the propagator approaches identity and the overlap decay is dominated by the perturbation operator,  $O(n) = |\langle \exp(-i\delta V) \rangle|^2 + O(\delta^2 \tau^2 n^2)$ . This behavior is demonstrated in Fig. 2 for  $\tau = 0.001$  and  $\tau = 0.01$  and with  $\delta = 0.3$ . For larger values of  $\tau$ , the fidelity decay under  $U_{\text{GUE}}$  obeys the FGR with the RMT rate (9).

The important point is that for sufficiently large  $\tau$  the eigenphases of  $U_{\text{GUE}}$  become spread pseudorandomly in the interval  $[0, 2\pi)$ . Under these conditions, the eigenphases of the map  $U_{\text{GUE}}$  exhibit the Poissonian spectral fluctuations that are characteristic of classically integrable (time-periodic) systems. We checked the nearest-neighbor spacing distribution of  $U_{\text{GUE}}$  and found that for  $\tau = 100$  the statistics are in excellent agreement with the Poissonian distribution  $P(s) \propto \exp(-s)$  (see inset of Fig. 2). However, the eigenvectors of  $U_{\text{GUE}}$  are random (by construction) and independent of  $\tau$  (for finite  $\tau$ ). From these observations, it is clear that the presence of Wigner-Dyson spectral fluctuations in the implemented  $U$ , which comprises the only basis-independent criterion of quantum chaos, is not actually necessary for the onset of exponential (FGR) decay at the rate (7). This suggests that it is the RMT statistics of the eigenvectors of  $U$  that lead to the FGR decay with the RMT rate.

We next consider the fidelity decay for the quantum kicked top, which is an exemplary dynamical model of quantum chaos [20,22]. The kicked top is a unitary map  $U_{\text{QKT}} = \exp(-i\pi J_y/2) \exp(-ikJ_z^2/j)$  acting on the Hilbert space of dimension  $N = 2j + 1$  associated with an irreducible representation of the angular momentum operator  $\vec{J}$ . In previous fidelity decay and LDOS studies, the choice of perturbation has usually been tied to a physical coordinate of the system  $U$ . We first follow this

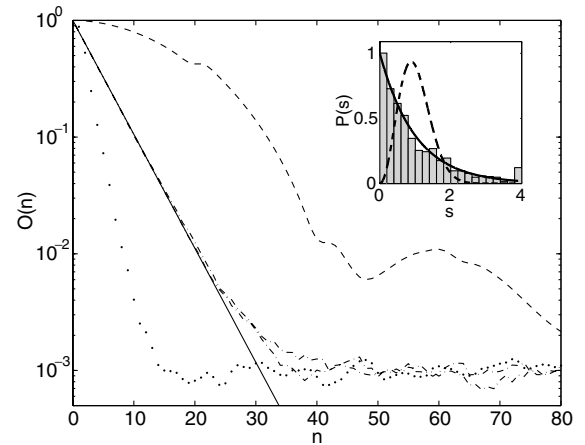


FIG. 2. Fidelity decay for  $U_{\text{GUE}}$  averaged over 50 computational basis states with  $\delta = 0.3$  for  $\tau = 0.001$  (dashed line),  $\tau = 0.01$  (dotted line), and  $\tau = 0.1$  and 100 (chain lines), compared to the FGR/RMT prediction (solid line). Inset: Spacing distribution for  $U_{\text{GUE}}$  for  $\tau = 100$  compared to Poissonian (solid line) and Wigner-Dyson (dashed line) distributions.

convention and identify the eigenbasis of the perturbation (5) with the eigenbasis  $|m_j\rangle$  of the system coordinate  $J_z$  (where  $m_j = \{j, \dots, -j\}$ ). In Fig. 3, we compare the fidelity decay for the chaotic and regular regimes of the kicked top for averages over 50 initial computational basis states. The fidelity decay for the chaotic top ( $k = 12$ ) is well described by the FGR prediction (4) and the RMT rate (9), whereas the regular top ( $k = 1$ ) shows a slower *nonexponential* decay rate. Similarly, if we associate the perturbation eigenbasis with the basis of the  $J_y$  coordinate, the fidelity decay for the chaotic top remains in agreement with the RMT rate and the regular top again exhibits nonexponential decay, though in this case with a *faster* decay than the RMT rate (9). However, we now demonstrate that an exponential decay at the RMT rate arises even for the regular kicked top when the qubit perturbation (5) is diagonal in a generic basis relative to the eigenbasis of  $U_{\text{QKT}}$ . Specifically, we leave the perturbation eigenvalue spectrum unchanged but set

$$U_p = T[\Pi_j \exp(-i\delta\sigma_z^j/2)]T^{-1}, \quad (10)$$

where  $T$  is drawn from CUE. As demonstrated in the inset of Fig. 3, under this type of perturbation the fidelity decay for the regular top is indistinguishable from that of the chaotic top and is very accurately described by the FGR at the RMT rate (9).

The sensitive dependence of the type of fidelity decay on the eigenbasis of the applied perturbation suggests a close connection with the basis dependence of the eigenvector statistics of classically regular quantum models. Expressed in a generic quantum basis, the eigenvectors of any quantized classical system  $U$  will have randomly (Gaussian) distributed components. In contrast, *in the eigenbases of the system coordinates* the components of

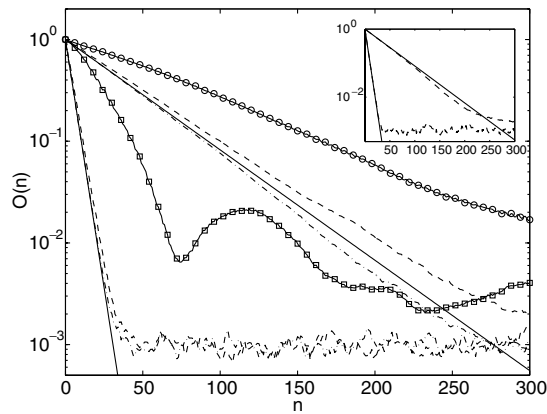


FIG. 3. Decay of  $O(n)$  for the kicked top in the chaotic regime ( $k = 12$ ) averaged over 50 computational basis states, with the perturbation eigenbasis mapped to the eigenbases of  $J_z$  (dashed lines) and  $J_y$  (chain lines) compared to the FGR decay (solid lines) at the RMT rate (9) for  $\delta = (0.1, 0.3)$  (top to bottom). Lines with circles and squares are for the regular kicked top ( $k = 1$ ) with the perturbation eigenbasis tied to the  $J_z$  and  $J_y$  coordinate bases, respectively (for  $\delta = 0.1$ ). Inset: Average fidelity decay for regular kicked top ( $k = 1$ ), when the qubit perturbation is in a random eigenbasis (10) and with  $\delta = (0.1, 0.3)$  (dashed lines), compared to the FGR/RMT rate (solid lines).

classically chaotic and integrable systems are known to be different [8], with the former Gaussian distributed and the latter exhibiting substantial deviation from the canonical Gaussian distribution. In light of this connection, in the case of quantized classical models we infer that exponential (nonexponential) fidelity decay can be correlated with the presence (absence) of characteristic RMT spectral fluctuations in the unperturbed system *provided that* the applied perturbation commutes with a system coordinate.

In summary, we have shown that the fidelity decay may be measured efficiently on a QIP. In the case of random unitary and Hermitian matrices, as well as a classically chaotic dynamical model, we have shown that the fidelity decays exponentially with a characteristic rate given precisely by the variance of the perturbation's eigenspectrum. The occurrence of the exponential decay is not directly dependent on the Wigner-Dyson fluctuations of the unperturbed spectrum, but does depend sensitively on the RMT statistics of the system eigenvectors in the eigenbasis of the applied perturbation. Hence, the fidelity decay for both classically regular and chaotic dynamical systems is given by the FGR under all but a small subset of unitary perturbation operators. In the case of classical models, we conclude that the fidelity decay provides a reliable indicator of RMT statistics (quantum chaos) in

the unperturbed system only when the applied perturbation is restricted to the subset of perturbations that commute with a classical coordinate.

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