A Class of Exact Classical Solutions to String Theory

A. A. Coley
Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia
(Received 11 September 2002; published 31 December 2002)

We show that the recently obtained class of spacetimes for which all of the scalar curvature invariants vanish (which can be regarded as generalizations of pp-wave spacetimes) are exact solutions in string theory to all perturbative orders in the string tension scale. As a result the spectrum of the theory can be explicitly obtained, and these spacetimes are expected to provide some hints for the study of superstrings on more general backgrounds. Since these Lorentzian spacetimes suffer no quantum corrections to all loop orders they may also offer insights into quantum gravity.

DOI: 10.1103/PhysRevLett.89.281601 PACS numbers: 11.25.–w, 04.20.Jb, 11.15.Kc

It is known that all of the scalar curvature invariants vanish in pp-wave spacetimes [1] (see also [2,3]). It has subsequently been argued that pp-wave spacetimes are exact solutions in string theory (to all perturbative orders in the string tension) [4,5]. In this Letter we shall show that this is true for a wide class of spacetimes (in addition to the pp-wave spacetimes), a result that has broad and important implications.

In a recent paper it was proven that in pseudo-Riemannian or Lorentzian spacetimes all of the scalar invariants constructed from the Riemann tensor and its covariant derivatives are zero if and only if the spacetime is of Petrov-type III, N or O, all eigenvalues of the Ricci tensor are zero, and the common multiple null eigenvector of the Weyl and Ricci tensors is shearfree, irrotational, geodesic, and expansion-free (SIGE) [6]; we shall refer to these spacetimes as vanishing scalar invariant (VSI) spacetimes in what follows for brevity. Utilizing a complex null tetrad in the Newman-Penrose (NP) formalism it was shown that for Petrov types III and N the repeated null vector of the Weyl tensor l^α is SIGE (i.e., the NP coefficients κ, σ, and ρ are zero), and the Ricci tensor has the form

$$R_{\alpha\beta} = -2\Phi_{22} l_\alpha l_\beta + 4\Phi_{21} l_{(\alpha} m_{\beta)} + 4\Phi_{12} l_{(\alpha} \bar{m}_{\beta)}, \quad (1)$$

in terms of the nonzero Ricci components Φ_{ij}. For Petrov-type 0, the Weyl tensor vanishes and so it suffices that the Ricci tensor has the form (1), where the corresponding vector field l^α is again SIGE.

All of these spacetimes belong to Kundt's class, and hence the metric of these spacetimes can be expressed [7,8]

$$ds^2 = 2du[Hdu + dv + Wd\zeta + \bar{W}d\bar{\zeta}] - 2d\zeta d\bar{\zeta}, \quad (2)$$

where $H = H(u, v, \zeta, \bar{\zeta})$ and $W = W(u, v, \zeta, \bar{\zeta})$ $(P \equiv 1)$, and the null tetrad is

$$\begin{aligned} l &= \partial_v, \\ n &= \partial_u - (H + W\bar{W})\partial_v + (\bar{W}\partial_\zeta + W\partial_{\bar{\zeta}}), \\ m &= \partial_\zeta. \end{aligned} \quad (3)$$

Note that in local coordinates the repeated null Weyl eigenvector is given by $l = \partial_v$. If $\tau = 0$, the null congruence l is recurrent [6].

The metrics for all VSI spacetimes are displayed in [6]. For example, the metric of Plebański-Petrov(PP)-type O, Petrov-type III, and NP coefficient $\tau = 0$ is given by

$$W = W_0(u, \bar{\zeta}), \qquad H = \frac{1}{2} v (W_{0,\bar{\zeta}} + \bar{W}_{0,\zeta}) + h_0(u, \zeta, \bar{\zeta}), \quad (4)$$

where

$$\Phi_{22} = [h_{0,\zeta\bar{\zeta}} - \Re(W_0 W_{0,\bar{\zeta}\bar{\zeta}} + W_{0,u\bar{\zeta}} + W_{0,\bar{\zeta}}^2)], \quad (5)$$

and l is recurrent. The generalized pp-wave solutions are of Petrov-type N, PP-type O (so that the Ricci tensor has the form of null radiation) with $\tau = 0$, and admit a covariantly constant null vector field [1]. The vacuum spacetimes, which are obtained by setting $\Phi_{22} = 0$, are the well-known pp-wave spacetimes (or plane-fronted gravitational waves with parallel rays).

The Ricci tensor (1) has four vanishing eigenvalues, and the PP type is N for $\Phi_{12} \neq 0$ or O for $\Phi_{12} = 0$. It is known that the energy conditions are violated in the PP-type N models [7] and hence attention is usually concentrated on the more physically interesting PP-type O case, which in the nonvacuum case corresponds to a pure radiation [7,9] (although we should point out that spacetimes that violate the energy conditions are also of interest in current applications [10]).

The pp-wave spacetimes have a number of important physical applications. In particular, pp-wave spacetimes are exact solutions in string theory (to all perturbative orders in the string tension) [4,5]. We shall show that a wide range of VSI spacetimes (in addition to the pp-wave spacetimes) also have this property. Indeed, pp waves provide exact solutions of string theory [4,5], and type-IIB superstrings in this background were shown to be exactly solvable even in of the presence of the RR five-form field strength [11,12]. As a result the spectrum of the theory can be explicitly obtained, and the number of

 © 2002 The American Physical Society

potential exact vacua for string theory is increased. This work is expected to provide some hints for the study of superstrings on more general backgrounds.

The classical equations of motion for a metric in string theory can be expressed in terms of σ-model perturbation theory [13], through the Ricci tensor $R_{\mu\nu}$ and higher-order corrections in powers of the string tension scale α' and terms constructed from derivatives and higher powers of the Riemann curvature tensor [e.g., $\frac{1}{2}\alpha' R_{\mu\rho\sigma\lambda}R_{\nu}{}^{\rho\sigma\lambda}$]. It has long been known that vacuum pp-wave spacetimes are exact solutions to string theory to all orders in α', and this was explicitly generalized to nonvacuum (null radition) pp-wave solutions in [5]. The proof that all of the VSI spacetimes [6] are classical solutions of the string equations to all orders in σ-model perturbation theory [4] consists of showing that all higher-order correction terms vanish [4], and this follows immediately from the results of [6]. This can be demonstrated explicitly by direct calculation for the type III example (4) (see also [14]). It is perhaps surprising that such a wide class of VSI spacetimes, which contain a number of arbitrary functions, have this property.

A more geometrical derivation of this result follows from the fact that the only nonzero symmetric second-rank tensor covariantly constructed from scalar invariants and polynomials in the curvature and their covariant derivatives in VSI spacetimes is the Ricci tensor (which is proportional to $l_\mu l_\nu$ for PP-type O spacetimes), and hence all higher-order terms in the string equations of motion automatically vanish [5]. More importantly, it is possible to generalize this approach to include other bosonic massless fields of the string. For example, we can include a dilaton Φ and an antisymmetric (massless field) $H_{\mu\nu\rho}$. Let us assume that for VSI spacetimes

$$\Phi = \Phi(u, \zeta, \bar{\zeta}), \tag{6}$$

$$H_{\mu\nu\rho} = A_{ij}(\mu, \zeta, \bar{\zeta})\ell_{[\mu}\nabla_{\nu}x^i\nabla_{\rho]}x^j. \tag{7}$$

The field equations [13]

$$2(\nabla\Phi)^2 - \nabla^2\Phi - \frac{1}{12}H^2 = 0, \tag{8}$$

$$\nabla_\lambda H^\lambda_{\mu\nu} - 2(\nabla_\lambda\Phi)H^\lambda_{\mu\nu} = 0, \tag{9}$$

are then satisfied automatically to leading order in σ-model perturbation theory (i.e., to order α'). This is clearly evident for VSI spacetimes with $\tau = 0$ and with $\Phi = \Phi(u)$ and $A_{ij} = A_{ij}(u)$ and follows from the fact that ℓ is recurrent; this can be shown explicitly by direct calculation for the type III example (4) and is known to be true for the pp waves. In these spacetimes $H^2 = \nabla^2\Phi = (\nabla\Phi)^2 = 0$, and the only nontrivial field equation is then [13]

$$R_{\mu\nu} - \frac{1}{4}H_{\mu\rho\sigma}H_{\nu}{}^{\rho\sigma} - 2\nabla_\mu\nabla_\nu\Phi = 0.$$

For PP-type 0 spacetimes ($R_{\mu\nu} \propto \ell_\mu\ell_\nu$) this equation has only one nontrivial component which then constitutes a single differential equation for the functions $\Phi, H_{\mu\nu\rho}$ and the metric functions (e.g., for pp-wave spacetimes $\partial^2 H + \frac{1}{18}A_{ij}A^{ij} + 2\Phi'' = 0$ [5]). Solutions with more general forms for $\Phi, H_{\mu\nu\rho}$ [than Eqs. (6) and (7)] are possible for $\tau = 0$ (and $\tau \neq 0$ spacetimes) and the field equations reduce to an underdetermined set of differential equations with many arbitrary functions.

We can consider higher-order corrections in σ-model perturbation theory, which are of the form of second-rank tensors and scalars constructed from $\nabla_\mu\Phi$, $H_{\mu\nu\rho}$, the metric and their derivatives. Since at most two derivatives of Φ can appear in any second-rank tensor and given the form of the Riemann tensor, all terms constructed from more that two $H_{\mu\nu\rho}$'s and their derivatives and at least one Riemann tensor and one or more $\nabla_\mu\Phi$'s or $H_{\mu\nu\rho}$'s must vanish. Also, for appropriately chosen Φ and $H_{\mu\nu\rho}$, all terms of the form $(\nabla\ldots\nabla H)^2$ must vanish. This can again be shown by direct calculation for the type III example (4) and was proven for the pp waves in [5]. Therefore, there are solutions to string theory to all orders in σ-model perturbation theory. In addition, there are clearly a large number of arbitrary functions (and many more than in the pp-wave case) in this class of solutions still to be determined.

It has been noted that massive fields can also be included since their loop contributions can always be expanded in powers of derivatives (the result will again be polynomials in curvature which will vanish). In addition, it has been shown [5] that exact pp waves are exact solutions to string theory, even nonperturbatively. It is therefore plausible that a wide class of VSI solutions, which depends on a number of arbitrary functions, are exact solutions to string theory nonperturbatively and worthy of further investigation. In particular, the singularity structure of the VSI string theory spacetimes can be studied as in [5].

Solutions of classical field equations for which the counter terms required to regularize quantum fluctuations vanish are also of importance because they offer insights into the behavior of the full quantum theory. The coefficients of quantum corrections to Ricci flat solutions of Einstein's theory of gravity in four dimensions have been calculated up to two loops. In particular, a class of Ricci flat (vacuum) Lorentzian 4-metrics, which includes the pp-wave spacetimes and some special Petrov-type III or N spacetimes, have vanishing counter terms up to and including two loops [14]. Thus these Lorentzian metrics suffer no quantum corrections to all loop orders [15]. In view of the vanishing of all quantum corrections in these spacetimes, it is possible that the VSI metrics are of importance and merit further investigation.

I would like to thank R. Milson, V. Pravda, A. Pravdová, and R. Zalaletdinov for discussions, and V. Pravda and A. Pravdová for further helpful comments on the manuscript. This work was supported, in part, by a research grant from NSERC.

[1] P. Jordan, J. Ehlers, and W. Kundt, Abh. Akad. Wiss. Mainz, Math.-Nat. **77** (1960).
[2] S. Deser, J. Phys. A **8**, 1972 (1975).
[3] G.W. Gibbons, Commun. Math. Phys. **45**, 191 (1975).
[4] D. Amati and C. Klimcik, Phys. Lett. B **219**, 443 (1989); **210**, 92 (1988).
[5] G.T. Horowitz and A.R. Steif, Phys. Rev. Lett. **64**, 260 (1990); Phys. Rev. D **42**, 1950 (1990).
[6] V. Pravda, A. Pravdová, A. Coley, and R. Milson, Classical Quantum Gravity (to be published), gr-qc/0209024.
[7] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact Solutions of the Einstein Field Equations* (Cambridge University Press, Cambridge, 1980).
[8] W. Kundt, Z. Phys. **163**, 77 (1961).
[9] G.C. Joly and M.A.H. MacCallum, Classical Quantum Gravity **7**, 541 (1990).
[10] C. Barcelo and M. Visser, gr-qc/0205066.
[11] R.R. Metsaev, Nucl. Phys. **B625**, 70 (2002).
[12] R.R. Metsaev and A.A. Tseytlin, hep-th/0202109.
[13] C.G. Callan, D. Friedman, E.J. Martinec, and M.J. Perry, Nucl. Phys. **B262**, 593 (1985).
[14] V. Pravda, Classical Quantum Gravity **16**, 3321 (1999).
[15] G.W. Gibbons, Classical Quantum Gravity **16**, L71 (1999).