The Parts Determine the Whole in a Generic Pure Quantum State

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We show that almost every pure state of multiparty quantum systems (each of whose local Hilbert space has the same dimension) is completely determined by the state's reduced density matrices of a fraction of the parties; this fraction is less than about two-thirds of the parties for states of large numbers of parties. In other words, once the reduced states of this fraction of the parties have been specified, there is no further freedom in the state.

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It is natural to think that a pure quantum state of n parties, chosen at random, would have some multiparty entanglement of all possible types including what one might call irreducible n-party entanglement. Giving concrete, and quantifiable, meaning to this idea is a major goal in the foundations of quantum mechanics and quantum information theory which has yet to be achieved.

Nonetheless, it has been shown that not all entanglement of n parties can be reversibly transformed into two-party entanglement [1–3], and indeed [1,2] that for any n there are states which cannot be transformed reversibly into states of fewer than n parties. One might deduce from this that there is a notion of irreducible n-party entanglement, even though it has thus far eluded us as to how to measure the amount of it that is contained in any given n-party state. We note that the general situation is different from the case of two parties where the entropy of entanglement is essentially the unique measure of the bipartite entanglement of two-party pure states [4].

In this Letter, we give results which throw a surprising light on these issues. We consider the case of pure states of any number n of parties each of which has a d-dimensional Hilbert space. We show that for almost all such states (i.e., for generic states of this type), the reduced states of a fraction of the parties (at most about two-thirds for large n) uniquely specify the full quantum state of the n parties; there are no other states, *pure or mixed*, consistent with the given reduced states. In the language of [5], we may say that all the information in a generic n-party state is contained in the reduced states of a fraction of the parties. Expressed differently, the low order correlations uniquely determine the high order correlations.

An earlier paper [5] considered this question for pure states of three qubits. It was shown that in this case the three two-party reduced states uniquely determine the full three-party state for generic pure states of three parties. One may wonder whether this is an anomalous case arising from the low dimensionality of the system. We show here that, on the contrary, this general picture, namely, that the high order correlations are determined by lower order ones, is the rule for generic pure quantum states in finite dimensions.

It may be worth remarking how different this situation is from the case of classical probability distributions. For generic distributions of n random variables each taking d values—such distributions arise, for example, from making local von Neumann measurements on the quantum systems we are considering—it is not difficult to show that even the set of all the marginal distributions for $n - 1$ of the variables does not uniquely specify the full probability distribution. Thus, there is no obvious classical analogue of the property of pure quantum states that we present here. (Classical probability distributions are more analogous to *mixed* quantum states.)

In a different context—the many-electron systems of molecular physics—much progress has been made on the related problem of reconstructing n-electron density matrices (especially where $n = 3$ or 4) from the two-particle reduced state [6,7]. Although our problem is similar in spirit, in molecular physics it is fundamental that the particles are indistinguishable (fermions) and so the fact that the full quantum state is totally antisymmetric plays a key role. In this Letter, we deal with the usual context of quantum information theory: The particles are distinguishable and the full quantum state need have no particular symmetry under interchange of the particles.

The plan of this Letter is first to show there is a fraction α_U of the parties such that, given the reduced states of this fraction of the parties, the only state (pure or mixed) consistent with these reduced states is the original state (the subscript U is to denote the fact that this is an upper bound). These reduced states uniquely specify the state and all the information in the full state is already contained in the reduced states. This result, however, leaves open the possibility that perhaps the true proportion of parties whose reduced states uniquely specify the full state is much smaller; indeed it might grow like $\log n$ for example. The second part of the Letter shows that, in fact, the number of parties must grow linearly with n. We give a lower bound α_L for this proportion; it is about 18.9% for systems of n qubits and grows to 50% for systems of n d-level systems, as d becomes large.

We will first show that there is an upper bound α_U on the fraction of parties whose reduced states are sufficient to uniquely specify the full state of n parties. To this end, we first consider three parties A, B, C with Hilbert spaces whose dimensions are M, N, P, respectively. We will take $M \geq N + P - 1$ for reasons which will become clear.

Consider then an arbitrary pure state $|\xi\rangle$ of three parties A, B, and C. We can write $|\xi\rangle$ as

$$|\xi\rangle = \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{k=1}^{P} a_{ijk}|ijk\rangle; \tag{1}$$

the labels in the ket refer to systems A, B, and C in that order. We wish first to answer the following question: Under what conditions is $|\xi\rangle$ uniquely determined by its two-particle reduced states?

A state that agrees with $|\xi\rangle$ in its reduced states but is not equal to $|\xi\rangle$ is most likely going to be a mixed state. In order to allow for this possibility, it is helpful to imagine an environment E with which the system might be entangled, such that the whole system, system plus environment, is in a pure state $|\psi\rangle$. Let us first ask what form $|\psi\rangle$ must take in order to be consistent with the (generally mixed) state of the pair AB derived from $|\xi\rangle$.

The density matrix of this two-particle state is at most of rank P, being confined to the space spanned by the vectors $|v_1\rangle = \sum_{ij} a_{ij1}|ij\rangle, |v_2\rangle = \sum_{ij} a_{ij2}|ij\rangle, \ldots, |v_P\rangle = \sum_{ij} a_{ijP}|ij\rangle$. The state $|\psi\rangle$ must thus be a superposition of the form

$$|\psi\rangle = |v_1\rangle|E_1\rangle + |v_2\rangle|E_2\rangle + \cdots + |v_P\rangle|E_P\rangle, \tag{2}$$

$|E_1\rangle, \ldots, |E_P\rangle$ being states of the joint system CE. Moreover, if the states $|v_1\rangle, \ldots, |v_P\rangle$ are linearly independent—this will be the case for almost all states $|\xi\rangle$—then in order to get the correct density matrix when one traces over C and E, the P states $|E_1\rangle, \ldots, |E_P\rangle$ must be orthonormal. Expanding $|E_1\rangle, \ldots, |E_P\rangle$ in the standard basis $\{|1\rangle, \ldots, |P\rangle\}$ of particle C, we obtain the following form for $|\psi\rangle$:

$$|\psi\rangle = \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{k,l=1}^{P} a_{ijl}|ijk\rangle|e_{lk}\rangle. \tag{3}$$

The states $|e_{lk}\rangle$ are states of E alone. The orthonormality conditions on $|E_1\rangle, \ldots, |E_P\rangle$ become

$$\sum_k \langle e_{lk}|e_{l'k}\rangle = \delta_{ll'}. \tag{4}$$

Thus, in order to match the reduced state on AB, $|\psi\rangle$ must be of the form given in Eq. (3), and for a generic $|\xi\rangle$, the $|e_{lk}\rangle$ in this equation must satisfy Eq. (4).

Similarly, in order to match the reduced state on AC, we must have

$$|\psi\rangle = \sum_{i=1}^{M} \sum_{j,l=1}^{N} \sum_{k=1}^{P} a_{ilk}|ijk\rangle|f_{lj}\rangle, \tag{5}$$

where the states $|f_{lj}\rangle$ are states of E satisfying (for generic $|\xi\rangle$)

$$\sum_j \langle f_{lj}|f_{l'j}\rangle = \delta_{ll'}. \tag{6}$$

To match the reduced state on BC, we must have

$$|\psi\rangle = \sum_{i,l=1}^{M} \sum_{j=1}^{N} \sum_{k=1}^{P} a_{ljk}|ijk\rangle|g_{li}\rangle, \tag{7}$$

with (again for generic $|\xi\rangle$)

$$\sum_i \langle g_{li}|g_{l'i}\rangle = \delta_{ll'}. \tag{8}$$

We will proceed by deriving consequences of the two equations (3) and (5)—both of these expressions must describe the same state $|\psi\rangle$. Thus,

$$\sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{k,l=1}^{P} a_{ijl}|ijk\rangle|e_{lk}\rangle = \sum_{i=1}^{M} \sum_{j,r=1}^{N} \sum_{k=1}^{P} a_{irk}|ijk\rangle|f_{rj}\rangle. \tag{9}$$

We now consider specific terms in this equation. For example, consider the terms with $|i11\rangle$ in them with i fixed. They lead to M equations (one for each choice of i)

$$\sum_{l=1}^{P} a_{i1l}|e_{l1}\rangle = \sum_{r=1}^{N} a_{ir1}|f_{r1}\rangle. \tag{10}$$

It is helpful to rearrange these equations as M homogeneous equations in the $N + P - 1$ variables,

$$(|e_{11}\rangle - |f_{11}\rangle), |e_{21}\rangle, \ldots, |e_{P1}\rangle, |f_{21}\rangle, \ldots, |f_{N1}\rangle. \tag{11}$$

Let us take the case that $M \geq N + P - 1$. In this case, for generic values of the a_{ijk}, the only solutions are

$$|e_{11}\rangle = |f_{11}\rangle; \qquad |e_{l1}\rangle = 0 \quad \text{for } l \neq 1; \qquad |f_{r1}\rangle = 0 \quad \text{for } r \neq 1. \tag{12}$$

Note that the M equations do not involve all the a_{ijk}, and there is no reason for the associated determinant to be zero in general.

Now consider the equations with $|i12\rangle$ in them with i fixed. These are M equations

$$\sum_{l=1}^{P} a_{i1l}|e_{l2}\rangle = \sum_{r=1}^{N} a_{ir2}|f_{r1}\rangle. \tag{13}$$

Using the fact that the only nonzero $|f_{r1}\rangle$ is $|f_{11}\rangle$, we can rearrange (13) into M equations in the P variables

$$(|e_{22}\rangle - |f_{11}\rangle), |e_{12}\rangle, |e_{32}\rangle, \ldots, |e_{P2}\rangle. \tag{14}$$

These equations will have solutions

$$|e_{22}\rangle = |f_{11}\rangle; \qquad |e_{l2}\rangle = 0 \quad \text{for } l \neq 2, \tag{15}$$

since again the determinant will not be zero, in the generic case.

Proceeding in this way to use the equations in $|i1k\rangle$, $k = 1, \ldots, P$, we find eventually that

$$|e_{lk}\rangle = \delta_{lk}|e_{11}\rangle. \tag{16}$$

Thus,

$$|\psi\rangle = \sum_{ijkl} a_{ijl}|ijk\rangle|e_{lk}\rangle = \sum_{ijk} a_{ijk}|ijk\rangle|e_{11}\rangle. \tag{17}$$

In other words, the fact that $|\psi\rangle$ must be consistent with the reduced states of AB and AC forces it to be a tensor product of the original pure state with a state of the environment. We note that, in getting to this result, we have not needed to make use of the requirement that the full state be consistent with the reduced state for BC.

We may now use this three-party result to learn about n-party systems. For let $N = P = d^m$, $M = d^{(m+1)}$, so that there are a total of $(3m + 1)$ d-level systems (clearly $M > N + P - 1$). The reduced states of $(2m + 1)$ parties determine the full state. In other words, for large numbers of parties, the knowledge of the reduced states of roughly $\alpha_U = 2/3$ of the parties is sufficient to uniquely specify the full pure state.

We notice that we have made no use of the orthogonality conditions for the environment states. Indeed, in the case of three qubits, we were able to use this orthogonality to show that the three two-party reduced states uniquely specify the full state of three parties for generic pure three-qubit states. We thus expect that the orthogonality conditions will allow us to reduce the fraction α_U of parties whose reduced states are required to specify the full state.

It will also have been noticed that we have derived the above bound for n-party systems by requiring that the full state be consistent with only two of the very many reduced states of the full system. One may well imagine that requiring consistency with all the reduced states reduces this fraction. Perhaps the number of parties needed might be much less than two-thirds of the total. Indeed perhaps it might grow sublinearly with n. We now show that in fact the true number of parties cannot be much less than our upper bound $n\alpha_U$: The number must grow linearly with n, and indeed for n qubits it must be more than about $0.189n$ for large n. We do this by finding a lower bound α_L; one must know the reduced states of at least this fraction of the parties.

Let us first consider the case of qubits. We consider a fraction α of the total number of qubits n. We have in mind that all the reduced states of this fraction of qubits are known. The total number of parameters in this set of reduced states must certainly be as large at the number $2^{n+1} - 2$ of parameters in the pure states we could hope to reconstruct.

We now estimate how many independent parameters there are in the reduced states of $n\alpha$ of the particles. We use the Bloch parametrization which is valid for any state, pure as well as mixed; e.g., for three parties any density matrix may be written as

$$\begin{aligned}\rho_{ABC} = \tfrac{1}{8}(&1 \otimes 1 \otimes 1 + \alpha_i\sigma_i \otimes 1 \otimes 1 + \beta_i 1 \otimes \sigma_i \otimes 1 \\ &+ \gamma_i 1 \otimes 1 \otimes \sigma_i + R_{ij}\sigma_i \otimes \sigma_j \otimes 1 \\ &+ S_{ij}\sigma_i \otimes 1 \otimes \sigma_j + T_{ij} 1 \otimes \sigma_i \otimes \sigma_j \\ &+ Q_{ijk}\sigma_i \otimes \sigma_j \otimes \sigma_k),\end{aligned} \tag{18}$$

since the set of matrices $(1, \sigma_x, \sigma_y, \sigma_z)$ is a basis for the operators on $\mathbb{C}^2$ (the parameters α_i, β_i, etc. are real).

Thus, for n parties the total number of parameters in the reduced states of up to $n\alpha$ parties is

$$\sum_{r=1}^{n\alpha} \binom{n}{r} 3^r. \tag{19}$$

Now

$$\frac{\binom{n}{r-1}3^{r-1}}{\binom{n}{r}3^r} = \frac{r}{3(n-r+1)} \leq \frac{\alpha}{3(1-\alpha)}, \tag{20}$$

for $r \leq n\alpha$.

Thus,

$$\sum_{r=1}^{n\alpha} \binom{n}{r} 3^r \leq \binom{n}{n\alpha} 3^{n\alpha}\left(\frac{3-3\alpha}{3-4\alpha}\right), \tag{21}$$

summing the geometric progression to infinity (we need $\alpha < 3/4$, but it will be; see below).

Now we find a value of α for which this total number of terms in the reduced states is less than $2^{n+1} - 2$. We want

$$\binom{n}{n\alpha} 3^{n\alpha}\left(\frac{3-3\alpha}{3-4\alpha}\right) \leq 2^{n+1} - 2. \tag{22}$$

Thus, at leading order in n, we need

$$e^{nH(\alpha)+n\alpha\ln 3} \leq e^{n\ln 2}, \tag{23}$$

where $H(x) = -x\ln x - (1-x)\ln(1-x)$.

Numerically, we find the solution to

$$H(\alpha) + \alpha\ln 3 - \ln 2 = 0 \tag{24}$$

to be $\alpha \sim 0.189$. Thus, for α less than this, there are not enough parameters in the reduced states to account for the different pure states.

Thus, taking the upper and lower bounds together, we conclude that, for generic pure states of n qubits, the reduced states of somewhere between $0.189n$ and $2n/3$ of the qubits uniquely specify the full quantum state; differently put, the correlations among between 0.189 and 2/3 of the qubits specify uniquely the high order correlations (i.e., the correlations among more of the qubits). Using the analogue of the parametrization in

(18) for an n-party state, we see that the higher order tensors in the expression for the pure state are determined by the lower order ones; these higher order tensors may not be freely varied once the lower order tensors are specified.

For systems of n parties, each of which lives in a d-dimensional Hilbert space, the argument for the lower bound is the same as that which we used for qubits. Rather than the three Pauli matrices, we use $d^2 - 1$ matrices to span the space of traceless Hermitian operators. Thus, the condition (24) becomes

$$H(\alpha) + \alpha \ln(d^2 - 1) - \ln d = 0. \tag{25}$$

One finds that the lower bound for the fraction increases with increasing d; from 18.9% for qubits to $1/2$ for large d.

We believe that it will be valuable to find the exact fraction of parties whose reduced states determine the full state for general values of n and d. The states which are *not* generic are also interesting. In the case of three qubits [5], the nongeneric states are those which are locally equivalent to

$$a|111\rangle + b|222\rangle. \tag{26}$$

There are many states consistent with the two-party reduced states for these pure states. We conjecture that nongeneric states for general n and d will have special properties as far as their multiparticle entanglement is concerned.

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