Polarization of Superfluid Turbulence

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We show that normal-fluid eddies in turbulent helium II polarize the tangle of quantized vortex lines present in the flow, thus inducing superfluid vorticity patterns similar to the driving normal-fluid eddies. We also show that the polarization is effective over the entire inertial range. The results help explain the surprising analogies between classical and superfluid turbulence which have been observed recently.

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Our concern is the experimental evidence that turbulent helium II appears similar to classical turbulence [1]. For example, the temporal decay of helium II turbulence behind a towed grid is the same as that expected in an ordinary fluid [2]. Furthermore, if helium II is agitated by rotating propellers, the energy spectrum obeys the same classical Kolmogorov $k^{-5/3}$ dependence on the wave number k [3]. It was also found that if helium II is forced at high velocity along pipes and channels the same pressure drops [4] are detected which are observed in a classical liquid. Furthermore, when a sphere moves at high velocity in helium II, the same drag crisis [5] is measured that occurs if the fluid is air.

These apparently classical results are surprising because helium II is a quantum fluid. According to Landau's two-fluid theory, it consists of the intimate mixture of an inviscid superfluid component and a viscous normal-fluid component. The latter is similar to a classical Navier-Stokes fluid, so, when made turbulent, it consists of eddies of various sizes and strengths. On the contrary, quantum mechanics constrains the rotational flow of the superfluid to quantized vortex lines, each vortex with the same quantum of circulation $\Gamma = 9.97 \times 10^{-4}$ cm^2/sec. Unlike what happens in a classical Euler fluid, superfluid vortex lines can reconnect with each other [6]. They also interact with the normal fluid via a linear mutual friction force [7], and, when helium II is made turbulent, they form a disordered, apparently random tangle. Clearly the experimental results described above call for an explanation in terms of the basic physical ingredients of the problem (normal-fluid eddies and superfluid vortices) and their interaction. What is remarkable is that these classical aspects of helium II turbulence are observed to be independent of temperature, whereas the relative proportion of superfluid and normal fluid is a strong function of temperature.

Years ago it was suggested [8] that vortex reconnections create an effective eddy viscosity which should make the superfluid similar to the normal fluid. It has been shown recently that vortex reconnections turn part of the superfluid's kinetic energy into sound energy (directly as bursts [9] and indirectly via Kelvin wave radiation [10]). There is also some numerical evidence from two different methods [11,12] that the superfluid energy spectrum obeys the $k^{-5/3}$ law over a short range, although better numerical resolution is needed to determine this power law under more realistic flow conditions. If both normal fluid and superfluid in isolation obey Kolmogorov's law, the key question is what happens when we take into account the mutual friction coupling the two fluids together.

Vinen [13] argued that at spatial scales larger than the average separation δ between the quantized vortex lines, the superfluid and normal fluid should be coupled by a small degree of polarization of the almost random tangle of superfluid vortex lines. The polarization should correlate the superfluid and normal-fluid velocity fields. If that is the case, on these scales helium II behaves as a single fluid of density $\rho = \rho_s + \rho_n$, consistently with the experiments (ρ_s and ρ_n are the superfluid and normal-fluid density, respectively).

The aim of this Letter is to support Vinen's argument with quantitative evidence of polarization. First, we shall introduce some simple models which, although very idealized, capture the essential physics of polarization. Second, we shall look for evidence of polarization by direct numerical simulation.

Our first model is concerned with the reaction of superfluid vortices to a normal-fluid shear. Consider a row of point vortices of alternating circulation $\pm\Gamma$ initially set along the x axis at distance δ from each other. We assume that the normal fluid is $\mathbf{v}_n = V_n \cos(ky)\hat{\mathbf{x}}$. The governing equation of motion of a vortex point is [14]

$$\frac{dy}{dt} = \pm\alpha V_n \cos(ky), \qquad (1)$$

where α is a known temperature dependent friction coefficient [7]. The solution of Eq. (1) corresponding to the initial condition $y(0) = 0$ is

$$y(t) = \frac{2}{k}\left(-\frac{\pi}{4} + \tan^{-1}(e^{\pm\alpha k V_n t})\right). \qquad (2)$$

Given enough time ($t \to \infty$), positive and negative vortices will reach stable locations $y_\infty = \pm\pi/2k$, respectively. In a turbulent normal flow, however, the shear does not last longer than the few times the turnover time $\tau \approx 1/\omega_n \approx 1/kV_n$. Since α is small (it ranges from 0.037 at $T = 1.3$ K to 0.35 at $T = 2.16$ K), we have $y(\tau) \approx \pm b$ where $b = \alpha/k$. Within the lifetime of the shear we have thus created a separation $2b$ between positive and negative vortices; that is to say, we have polarized the initial configuration. The velocity of this Karman-vortex street is approximately [15] $V_s \approx \pi\Gamma/2\delta$ for $k\delta \ll 1$ in the direction along the x axis where the normal fluid (which induced the polarization in the first place) is stronger. The result suggests that it is not necessary to create extra vortex lines to generate a superfluid pattern that mimics the normal-fluid one—rearranging existing vortices is enough. We also notice that the induced polarization is proportional to α.

Our second model is concerned with the expansion of favorably oriented superfluid vorticity. We think of the superfluid vortex tangle as a collection of vortex rings of radius approximately equal to the average separation of vortices in the tangle, $R_0 \approx \delta$. We assume for simplicity that the rings are on the x, y plane with equal numbers of rings oriented in the $\pm z$ directions. Depending on whether they have positive or negative orientation, the rings move along $\pm z$ with self-induced speed given by

$$V_{R_0} = \frac{\Gamma\mathcal{L}}{4\pi R_0}, \tag{3}$$

where $\mathcal{L} = \ln(8R_0/a_0) - 1/2$ is a slowly varying term and $a_0 \approx 10^{-8}$ cm is the vortex core radius. Now we apply a normal-fluid velocity V_n in the z direction. The radius R of a ring is determined by [7]

$$\frac{dR}{dt} = \frac{\gamma}{\rho_s\Gamma}(V_n - V_R), \tag{4}$$

where γ is a known friction coefficient and $\gamma/\rho_s\Gamma \approx \alpha$ at almost all temperatures of interest. Equation (4) shows that V_n selectively changes radius and velocity of vortex rings moving in opposite directions. A ring which grows (shrinks) by an amount $\delta R = \alpha\delta t(V_n - V_{R_0})$ in time δt slows down (speeds up) by an amount $\delta V_R = V_{R_0}\delta R/R_0$. In this way a superflow is induced in the same direction of the normal fluid which induced the polarization in the first place. A simple estimate of the spatial averaged magnitude of this superflow yields [16] $V_s \approx 3V_{R_0}\delta R/R_0$.

Our third model is concerned with the rotation of existing superfluid vorticity. We represent a superfluid vortex line as a straight segment pointing away from the origin and study how its orientation is changed by a normal-fluid rotation about the z axis. Using spherical coordinates (r, θ, ϕ), we assume that the vortex is initially in the plane $\theta = \pi/2$. The normal fluid's velocity is $\mathbf{v}_n = (0, 0, \Omega r \sin\theta)$, and the motion of the vortex segment is determined by [17] $d\theta/dt = -\alpha\Omega\sin(\theta)$, $dr/dt = 0$, and $d\phi/dt = 0$. The solution is $\theta(t) = 2\tan^{-1}(e^{-\alpha\Omega t})$, with r and ϕ constant. Given enough time, the vortex segment will align along the direction of the normal-fluid rotation ($\theta \to 0$ for $t \to \infty$), but the lifetime τ of the eddy is only of the order of $\tau \approx 1/\Omega$, so the vortex can turn only to the angle $\theta(\tau) \approx \pi/2 - \alpha$.

Despite the smallness of the angle, the effect is sufficient to create a net polarization of the tangle in the direction of the normal fluid's rotation, provided that there are enough vortices. The following argument shows how this is possible. The normal fluid is like a classical viscous Navier-Stokes fluid, and, if left to itself, its spectrum E_k would obey Kolmogorov's law

$$E_k = C\epsilon^{2/3}k^{-5/3}. \tag{5}$$

Equation (5) is valid in the inertial range $1/\ell_0 < k < 1/\eta$ in which big eddies break up into smaller eddies, transferring energy to higher and higher wave numbers without viscosity playing a role. Here k is the magnitude of the three dimensional wave vector, ϵ is the rate of energy dissipation per unit mass, C is a constant of order unity, ℓ_0 is the integral length scale (the scale at which energy is fed into the energy cascade), and η is the Kolmogorov scale at which kinetic energy is dissipated by the action of viscosity. In reality the normal fluid is not alone but is forced by the quantized vortex filaments. We know little of the effects of this forcing (it has been studied only for very simple geometries [18]) so, for lack of further information, we assume that the classical relation Eq. (5) is valid for the normal fluid.

The quantity which is often used to describe the intensity of the superfluid vortex tangle is the vortex line density L (length Λ of vortex line per volume $\mathcal{V}$) because it is easily measured by detecting the attenuation of second sound. From L one infers the average separation between vortices, $\delta \approx L^{-1/2}$. The quantity ΓL can be interpreted as the total rms vorticity of the superfluid. Note that the net amount of superfluid vorticity in a particular direction can be much less than ΓL (even zero, if the tangle is randomly oriented).

The key question is whether, as a result of mutual friction, sufficient quantized vortex lines can reorient themselves within a normal-fluid eddy of wave number k so that the resulting net superfluid vorticity matches the vorticity ω_k of that eddy. The process must take place in a time scale shorter than the typical lifetime of the eddy, which is of the order of a few times the turnover time $1/\omega_k$. At this point we use the result of the third model, for which the initial condition $\theta(0) = \pi/2$ represents the average case. If the initial orientation of the vortex is toward the origin rather than away from it, then the vortex segment turns to $\theta(\tau) \approx \pi/2 + \alpha$ rather than $\pi/2 - \alpha$ but still contributes to positive vorticity in the z direction. Therefore in the time $1/\omega_k$, reordering of existing vortex lines creates a net superfluid vorticity ω_s of the order of $\alpha L\Gamma/3$ in the direction of the vorticity ω_k of the driving normal-fluid eddy of wave number k. Since ω_k is approximately $\omega_k = \sqrt{k^3E_k}$, we have $\omega_k = C^{1/2}\epsilon^{1/3}k^{2/3}$. Matching ω_s and ω_k would then require

$$\tfrac{1}{3}\alpha\Gamma L \geq C^{1/2}\epsilon^{1/3}k^{2/3}. \quad (6)$$

The normal-fluid vorticity increases with k and is concentrated at the smallest scale ($k \approx 1/\eta$), so a vortex tangle with a given value of L may satisfy the above equation up to only a certain critical wave number k_c. Substituting $\epsilon = \nu_n^3/\eta^4$ where ν_n is the normal fluid's kinematic viscosity (the viscosity of helium II divided by ρ_n), we obtain $(\delta/\eta) = C^{-1/4}(\alpha/3)^{1/2}(\Gamma/\nu_n)^{1/2} \times (\eta k_c)^{-1/3}$. If $k_c \approx 1/\eta$, then

$$\frac{\delta}{\eta} = C^{-1/4}\left(\frac{\alpha}{3}\right)^{1/2}\left(\frac{\Gamma}{\nu_n}\right)^{1/2}. \quad (7)$$

In the temperature range of experimental interest Γ/ν_n ranges from 0.43 at $T = 1.3$ K to 5.86 at $T = 2.15$ K, so $\delta/\eta = \mathcal{O}(1)$ and we conclude that matching of the two vorticities ($k_c \approx 1/\eta$) is possible throughout the inertial range.

Because of the computational cost, it is difficult to compute numerically vortex tangles dense enough to cover the range $k < 1/\ell$. To make progress in the problem and confirm the above arguments we study the reaction of the superfluid vortex tangle to a single scale *ABC* normal flow [19] given by $\mathbf{v}_n = [A\sin(kz) + C\cos(ky); B\sin(kx) + A\cos(kz); C\sin(ky) + B\cos(kx)]$ where $k = 2\pi/\lambda$ is the wave number, λ is the wavelength, and A, B, and C are parameters. *ABC* flows are solutions of the Euler equation and the forced Navier-Stokes equation and have been used as an idealized model of eddies in fluid dynamics, magnetohydrodynamics, and superfluid hydrodynamics [20]. For the sake of simplicity we take $A = B = C$ and $\lambda = 1$.

We represent a superfluid vortex filament as a space curve $\mathbf{s} = \mathbf{s}(\xi, t)$ where ξ is arclength and t is time. Neglecting a small transverse friction coefficient, the curve moves with velocity

$$\frac{d\mathbf{s}}{dt} = \mathbf{v}_{si} + \alpha\mathbf{s}' \times (\mathbf{v}_n - \mathbf{v}_{si}), \quad (8)$$

where $\mathbf{s}' = d\mathbf{s}/d\xi$ and the self-induced velocity $\mathbf{v}_{si}$ is given by the Biot-Savart integral

$$\mathbf{v}_{si} = \frac{\Gamma}{4\pi}\int\frac{(\mathbf{r} - \mathbf{s}) \times \mathbf{dr}}{|\mathbf{r} - \mathbf{s}|^3}. \quad (9)$$

The calculation is performed in a cubic box of volume $\mathcal{V} = 1$ cm^3 with periodic boundary conditions. The numerical technique is standard [21], and the details of our algorithm, including how to perform vortex reconnections, have been published elsewhere [20].

We start the calculation with $N = 50$ superfluid vortex rings set at random positions and orientation and integrate in time at a variety of temperatures ($\alpha = 0.1$, 0.5, and 1.0) and normal fluid's velocities ($A = 0.01$, 0.1, 1.0, and 10.0 cm/sec). The vortex length ($\Lambda = 76.8$ cm at $t = 0$) increases or decreases depending on whether the *ABC* flow is strong enough to feed energy into the vortices via instabilities of vortex waves (for example, for $\alpha = 1.0$, the final length Λ is as high as 781.3 cm at $A = 10.0$ cm/sec, and as low as 56.67 cm at $A = 0.01$ cm/sec). The rings interact with each other and with the normal fluid, get distorted, reconnect, and soon an apparently random tangle is formed (see Fig. 1).

The quantity $\langle\cos(\theta)\rangle = \langle\mathbf{s}' \cdot \hat{\boldsymbol{\omega}}_n\rangle$, which we monitor during the evolution, gives us the tangle-averaged projection of the local tangent to a vortex in the direction of the local normal-fluid vorticity, $\hat{\boldsymbol{\omega}}_n = (1/\omega_n)\boldsymbol{\omega}_n$, where $\boldsymbol{\omega}_n = \nabla \times \mathbf{v}_n$. At $t = 0$ $\langle\cos(\theta)\rangle = 0$ due to the random nature of the initial state, and it is apparent from Fig. 2 that $\langle\cos(\theta)\rangle$ increases with time, no matter whether Λ decreases or increases.

The results are analyzed in Fig. 3. From the simple models described above we expect that the polarization induced by the normal-fluid vorticity is proportional to α. We also know from the discussion above that we should restrict the analysis to times $t < \tau$ where $\tau = 1/\omega_n$ with $\omega_n = \sqrt{3}Ak$ is the lifetime of the normal-fluid eddy, which we assume to be the same as the eddy's turnover time. It is apparent from the figure that, no matter whether the tangle grows or decays, approximately the same polarization takes place for $t/\tau < 1$.

In conclusion, we have put the theory of superfluid turbulence on firmer ground. Using simple models which capture the essential physical mechanisms of polarization and then a numerical simulation, we have shown that, within the lifetime of a normal-fluid eddy of wave

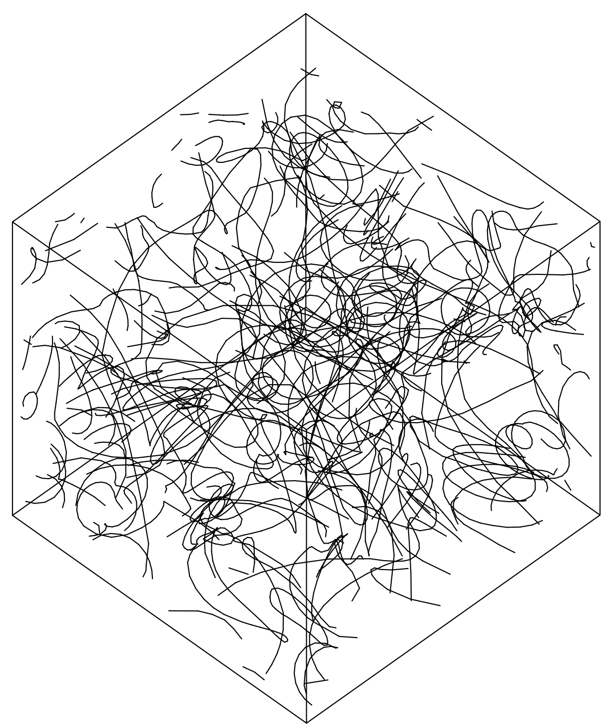

FIG. 1. Vortex configuration at $t = 0.123$ sec for $\alpha = 0.5$ and $A = 1.0$ cm/sec.

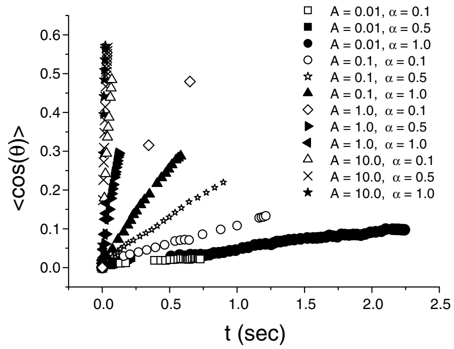

FIG. 2. Average polarization $\langle\cos(\theta)\rangle$ versus time t computed for different values of A and α.

number k, superfluid vortex lines can rearrange themselves so that the superfluid vorticity and the normal-fluid vorticity are aligned. Provided that enough vortex lines are present, vorticity matching should take place over the entire inertial range, up to wave numbers k of the order of $1/\ell$.

Our result has theoretical and experimental implications. Numerical simulations of vortex lines driven by normal-fluid turbulence [22] show a k^{-1} superfluid energy spectrum in the region $k \geq 1/\delta$. More intense (hence computationally expensive) vortex tangles should be investigated to explore the region $k \ll 1/\delta$ where we predict the classical $k^{-5/3}$ dependence observed without normal fluid [11,12]. Another important issue which must be investigated is the nonlinear saturation of the polarization process. On the experimental side, our result supports the use of helium II to study classical turbulence. This has been done recently by Skrbek *et al.* [23] who exploited the physical properties of liquid helium to study the decay of vorticity on an unprecedented wide range of scales.

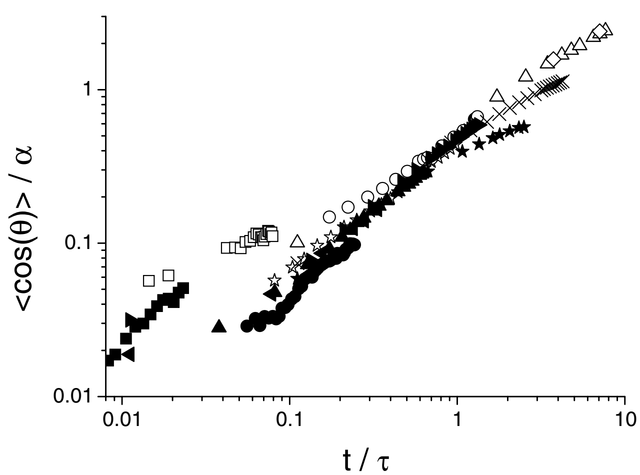

FIG. 3. $\langle\cos(\theta)\rangle/\alpha$ versus scaled time t/τ where τ is the eddy's lifetime (same symbols as in Fig. 1).

We are indebted to W. F. Vinen who suggested we study some of the above described models of the interaction between normal-fluid and quantized vortex lines.

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[14] Equation (1) follows from Eq. (8) in the absence of $\mathbf{v}_{si}$.

[15] We estimate the velocity of the double row of vortices at $(ma, b/2)$ and $[(m + 1/2)a, -b/2]$ (where m is an integer and $a = 2\delta$) at the origin and obtain $V_s = (\pi\Gamma/a)\sinh(\pi b/a)/[\cosh(\pi b/a) - 1] \approx \pi\Gamma/2\delta$.

[16] Let n be the number of positive and negative rings. Their areas and velocities are, respectively, $A^{\pm} = \pi R_0^2$ and $V^{\pm} = \pm V_{R_0}$. At $t = 0$ the average velocity is $V_s = [n\pi R_0^2 V_{R_0} - n\pi R_0^2 V_{R_0}]/[n\pi R_0^2 + n\pi R_0^2] = 0$. At time $t = \delta t$ we have $A^{\pm} = \pi(R_0 \pm \delta R)^2$ and $V^{\pm} = \pm(V_{R_0} \mp \delta V_R)$, so $V_s = [n\pi(R_0 + \delta R)^2(V_{R_0} - \delta V_R) - n(R_0 - \delta R)(V_{R_0} + \delta V_R)]/[n\pi(R_0 + \delta R)^2 + n\pi(R_0 - \delta R)^2] \approx 3V_{R_0}\delta R/R_0$.

[17] Again, this equation follows from Eq. (8) neglecting the self-induced motion which is zero for a straight vortex.

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