Eliminating Islands in High-Pressure Free-Boundary Stellarator Magnetohydrodynamic Equilibrium Solutions

S. R. Hudson,[1] D. A. Monticello,[1] A. H. Reiman,[1] A. H. Boozer,[2] D. J. Strickler,[3] S. P. Hirshman,[3] and M. C. Zarnstorff[1]
[1]*Princeton Plasma Physics Laboratory, P.O. Box 451, Princeton, New Jersey 08543*
[2]*Columbia University, New York, New York 10027*
[3]*Oak Ridge National Laboratory, P.O. Box 2009, Oak Ridge, Tennessee 37831*
(Received 3 June 2002; published 20 December 2002)

Magnetic islands in free-boundary stellarator equilibria are suppressed using a procedure that iterates the plasma equilibrium equations and, at each iteration, adjusts the coil geometry to cancel resonant fields produced by the plasma. The coils are constrained to satisfy certain measures of engineering acceptability and the plasma is constrained to ensure kink stability. As the iterations continue, the coil geometry and the plasma simultaneously converge to an equilibrium in which the island content is negligible. The method is applied with success to a candidate plasma and coil design for the National Compact Stellarator Experiment [Phys. Plasmas **8**, 2083 (2001)].

DOI: 10.1103/PhysRevLett.89.275003 PACS numbers: 52.55.Hc

Introduction.—The magnetic field lines of toroidal plasma confinement devices, such as stellarators [1], are $1\frac{1}{2}$-dimensional Hamiltonian systems and magnetic flux surfaces are the analog of constant action surfaces [2]. This may be seen by noting that in arbitrary toroidal coordinates (r, θ, ζ) any vector, in particular, the magnetic vector potential, may be written $\mathbf{A} = \psi\nabla\theta - \chi\nabla\zeta + \nabla g$, where ψ, χ, and g are functions of (r, θ, ζ): from which $\mathbf{B} = \nabla\psi \times \nabla\theta + \nabla\zeta \times \nabla\chi$. Using the toroidal angle ζ as the independent (time) coordinate, and considering $\chi = \chi(\psi, \theta, \zeta)$, the magnetic field line flow equations may be recast in a form identical to Hamilton's equations: $d_\zeta\theta = \partial_\psi\chi$ and $d_\zeta\psi = -\partial_\theta\chi$.

For magnetohydrodynamic (MHD) equilibrium, the pressure gradient force must balance the Lorentz force $\nabla p = \mathbf{J} \times \mathbf{B}$, which requires $\mathbf{B} \cdot \nabla p = 0$. In regions where $|\nabla p| \neq 0$, the field is integrable, $\mathbf{B} \cdot \nabla\psi = 0$, and action-angle coordinates exist $\chi = \chi(\psi)$. In this context, action-angle coordinates are called magnetic coordinates and $\mathbf{B} \cdot \nabla f = (\mathbf{B} \cdot \nabla\zeta)(\iota\partial_\theta + \partial_\zeta)f$ for an arbitrary function f and $\iota = \partial_\psi\chi$ is called the rotational transform.

Integrable $1\frac{1}{2}$-dimensional Hamiltonians naturally occur only in systems with a continuous symmetry, and stellarators have no continuous symmetry. Integrability can be studied by perturbing an integrable field $\mathbf{B}_0$. Writing $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$ and $\psi = \psi_0 + \psi_1$, the perturbed system is integrable if $\mathbf{B}_0 \cdot \nabla\psi_1 + \mathbf{B}_1 \cdot \nabla\psi_0 = 0$. In magnetic coordinates, this becomes

$$\iota\frac{\partial\psi_1}{\partial\theta} + \frac{\partial\psi_1}{\partial\zeta} = -\frac{\mathbf{B}_1 \cdot \nabla\psi_0}{\mathbf{B}_0 \cdot \nabla\zeta}. \quad (1)$$

If this can be nontrivially solved for ψ_1, new magnetic coordinates exist and the perturbed state preserves integrability; however, the Fourier coefficients of ψ_1 are given by the Fourier coefficients of (B_1^ψ/B_0^ζ) divided by $(\iota m - n)$. At rational rotational-transform surfaces, $\iota = n/m$, a singularity exists and the perturbed state is nonintegrable. In the perturbed state, the rational surface splits to form an island of width $[(B_1^\psi/B_0^\zeta)_{mn}/\iota' m]^{1/2}$. Islands and the chaotic field lines caused by island overlap result in poor plasma confinement.

In an ideal MHD model of plasma perturbations, singular currents arise at the rational surfaces, both from δ-function currents that arise to guarantee that islands can neither be created nor destroyed and from singularities in the Pfirsch-Schlüter currents. The δ-function currents cannot exist in a plasma equilibrium consistent with finite resistivity, and nonzero $(B_1^\psi/B_0^\zeta)_{mn}$ and, thus, islands may exist. In the presence of islands, the Pfirsch-Schlüter currents are nonsingular [3]. Numerical codes that model such equilibria, such as the PIES code [4], must allow for the magnetic field to have islands. Note that there is an effect on the island width from the plasma currents in the island interior and near the separatrix (the apparent δ-function currents as seen by the exterior solution in a boundary layer analysis [5]), and this effect is included in the PIES code.

Changes in coil geometry will change $(B_1^\psi/B_0^\zeta)_{mn}$ and can reduce the magnitude of the islands and their associated stochastic regions. It may not be possible to completely eliminate all islands [6], but all that is required in practice is that the magnetic islands occupy less than a tolerable percentage of the plasma volume. Such a magnetic field is said to have "good-flux surfaces."

The construction of vacuum magnetic fields with good-flux surfaces is not trivial [7], but is simpler than when a plasma is present. The additional complexity arises from the modification of $(B_1^\psi/B_0^\zeta)_{mn}$ by the plasma currents, and the self-consistent solution requires that the plasma equilibrium field and the coil field combine to give a zero resonant component at the rational surfaces. Previous studies of finite pressure stellarator equilibria with islands have showed that the width of an island can depend on the magnitude of the plasma pressure and even go to zero, an effect called "self-healing" [8]. A recent article [9] showed that high-pressure *fixed*-boundary solutions may be constructed with good-flux surfaces.

Stellarators are designed to optimize both their physics properties (particle orbits, MHD stability, etc.) and the engineering of the coils. The optimizations rely on plasma equilibrium codes, and the fastest such codes presuppose perfect flux surfaces — the existence or size of magnetic islands cannot be addressed. The purpose of our study is to enforce good-flux surfaces by varying the shape of the coils while preserving the optimized properties of the plasma and the coils. Stellarator coils must balance the normal field B_n produced by the plasma currents on the plasma surface. Balancing B_n at each point on an arbitrary surface represents an infinite number of constraints and generically leads to singular coil currents. Fortunately, each resonant $(B_1^\psi/B_0^\zeta)_{mn}$ that must be controlled constrains the magnitude of only one spatial distribution of B_n on the plasma surface, and it is only this spatial distribution which must be nulled to eliminate the island.

The motivation for this work was the design of the National Compact Stellarator Experiment (NCSX) [10]. Features of this design make the enforcement of good-flux surfaces more difficult than in traditional stellarators [1]. NCSX is compact with a pronounced lack of geometric symmetry and has a large shear and transform per period, which produce multiple low-order resonances. In addition, NCSX is designed to operate with significant plasma current and at high plasma pressure (above 4% of the averaged magnetic energy), which means the rotational-transform profile and the shape of the magnetic surfaces are equilibrium dependent.

We found: (1) Adjustments to the coil shapes allow the enforcement of good-flux surfaces while maintaining optimized plasma and engineering properties of a particular NCSX equilibrium. (2) Coils obtained from healing a single reference configuration actually support many other optimized NCSX equilibria while maintaining good-flux surfaces. The second result indicates the primary issue with the NCSX flux surfaces is coil design and not the plasma equilibrium. The phenomenon of self-healing implies that this result is not generic, but the improved flux surfaces seen for a class of NCSX equilibria builds confidence that the method has practical as well as fundamental physical interest.

The free-boundary PIES code, which was used in the study, finds solutions by iterating the MHD equilibrium equations and has a representation of the magnetic field that accommodates magnetic islands and chaotic field lines. To suppress islands, the standard PIES algorithm is augmented so the coil geometry is altered at each iteration to cancel the resonant magnetic field components produced by plasma currents. The adjustment of the coils at each iteration allows the retention of the inherently nonlinear plasma response. To preserve the previous optimization of the coils and the plasma, changes in the coil geometry are constrained to preserve engineering constraints of minimum bend radius and coil-coil separation, as well as the plasma constraint of ideal kink stability. As the iterations continue, the coil geometry and the plasma equilibrium simultaneously converge to an island-free, stable plasma with buildable coils.

Method.—The total magnetic field is the sum of the magnetic field produced by the plasma, $\mathbf{B}_P$, and the magnetic field produced by the confining coils, $\mathbf{B}_C$, which is a function of a set of Fourier harmonics, $\boldsymbol{\xi}$, which describe the coil geometry, at the nth PIES iteration

$$\mathbf{B}^n = \mathbf{B}_P^n + \mathbf{B}_C(\boldsymbol{\xi}^n). \tag{2}$$

The initial plasma state is provided by the VMEC code [11], which imposes the artificial constraint that the plasma has nested flux surfaces, and the initial coil geometry is provided by the COILOPT code [12]. The method presented in this Letter removes the constraint of nested surfaces and allows the VMEC initialization to relax into an equilibrium, potentially with broken flux surfaces (islands), while making adjustments to the coil set to remove selected islands as they develop.

The PIES iterations solve for the plasma current $\mathbf{J}$ given $\mathbf{B}$ and given pressure profile p

$$\nabla p = \mathbf{J}^{n+1} \times \mathbf{B}^n. \tag{3}$$

A magnetic-differential equation $\mathbf{B} \cdot \nabla(J_\parallel/B) = \nabla \cdot J_\perp$ gives the parallel current which is solved using magnetic coordinates [13], and the current profile enters as an integration constant. PIES uses current profiles which are consistent with an Ohm's law with finite resistivity, thus eliminating the δ-function parallel currents. The PIES code allows the field topology to break up into islands and chaos. In the region interior to the islands, because of thermal and particle diffusion and in the absence of sources and sinks, the pressure is constant and thus the magnetic-differential equation for the Pfirsch-Schlüter currents need not be solved. Also, to be consistent with Ohm's and Faraday's laws in steady state, the current profile is flattened inside the island.

The plasma magnetic field is then solved, given $\mathbf{J}$, and blended to provide numerical stability:

$$\mathbf{J}^{n+1} = \nabla \times \mathbf{B}_P, \tag{4}$$

$$\mathbf{B}_P^{n+1} = \alpha \mathbf{B}_P^n + (1-\alpha)\mathbf{B}_P. \tag{5}$$

Typically, the blending parameter $\alpha = 0.99$ for NCSX style equilibria. The standard PIES algorithm makes no changes to the coil geometry and iterates through Eqs. (3)–(5) to calculate the free-boundary equilibrium for a given pressure profile and coil set.

The additional steps in the implementation of the coil healing are as follows. The total magnetic field $\bar{\mathbf{B}}$ is

$$\bar{\mathbf{B}} = \mathbf{B}_P^{n+1} + \mathbf{B}_C(\boldsymbol{\xi}^n). \tag{6}$$

We may consider $\bar{\mathbf{B}}$ as a *nearly* integrable field and that magnetic islands are caused by fields normal to and resonant with rational rotational-transform flux surfaces of a *nearby* integrable field.

A set of resonances that are to be suppressed is selected. The selection is determined by the rotational-transform profile. Islands associated with low-order rationals are typically the largest, but where the shear is small, high-order islands can easily overlap and result in chaotic field lines. A set of toroidal surfaces matching the selected resonances is constructed. Each such surface (a quadratic-flux-minimizing surface [14]) may be considered as a rational rotational-transform flux surface of an underlying integrable field [15], with each surface passing directly through its associated island chain and containing the stable and unstable periodic orbits. The construction of the quadratic-flux-minimizing surfaces provides an optimal magnetic coordinate system or, equivalently, an optimal nearby integrable magnetic field, and in these coordinates resonant perturbation harmonics are clearly identified. The method is computationally efficient as the quadratic-flux-minimizing surfaces are constructed exactly and only where required—at the rational rotational-transform surfaces where islands develop.

The amplitude of each of the N selected resonant field harmonics, denoted $\{\bar{B}_i{:}i = 1, N\}$, is calculated by Fourier decomposing the magnetic field normal to the quadratic-flux-minimizing surface. The Fourier decomposition is performed using an angle coordinate which corresponds to a magnetic coordinate angle of the underlying integrable field on that surface.

The COILOPT [12] code provides a convenient Fourier representation of the coil geometry and a set of M coil harmonics $\{\xi_j{:}j = 1, M\}$ is systematically varied to set $\bar{B}_i = 0$ using a Newton method. The coupling matrix, ∇B^n_{Cij}, is defined as the partial derivatives of the selected resonant harmonics of the coil magnetic field normal to the quadratic-flux-minimizing surface, which is updated every PIES iteration, with respect to the chosen coil harmonics and is calculated using finite differences. A multidimensional Newton method is applied to find the coil changes $\delta\xi_j$ that set $\bar{B}_i = 0$

$$-\bar{B}_i = \sum_j \nabla B^n_{Cij} \cdot \delta\xi^n_j. \qquad (7)$$

This equation is solved for the $\delta\xi_j$ in a few iterations by inverting the $N \times M$ matrix ∇B^n_{Cij} using singular-value decomposition [16] and the coil set is adjusted

$$\xi^{n+1}_j = \xi^n_j + \delta\xi^n_j, \qquad (8)$$

at every PIES *iteration,* such that resonant components of the combined plasma-coil field are eliminated. As the iterations proceed, the coil geometry and the plasma simultaneously converge to coil geometry-plasma solution with good-flux surfaces.

To be "buildable," the minimum coil curvature and coil-coil separation, for example, of the coils must exceed certain limits. Such constraints are calculated by the COILOPT code and the initial coil set, described by $\boldsymbol{\xi}^0$, is satisfactory from an engineering perspective. The healing algorithm is modified to preserve the minimum curvature and coil-coil separation by adding to the set of resonant fields to be eliminated the (appropriately weighted) differences in minimum curvature and coil-separation of the nth coil set, described by $\boldsymbol{\xi}^n$, from the initial coil set. This constrains the island-eliminating coil variations to lie in the null space of these measures of engineering acceptability. In a similar manner, the algorithm preserves kink stability. The VMEC initialization is kink stable, and kink stability is calculated with the TERPSICHORE code [17].

Application to NCSX.—The method is routinely applied to NCSX [10] candidate coil and plasma designs. NCSX is a proposed proof-of-principle device with three field periods, aspect ratio $A = 4.4$, major radius $R = 1.4$ m, and magnetic field $B = 1.7$ T. The stellarator symmetric coil design consists of 18 modular coils (three distinct coil types), 18 toroidal field coils, and six pairs of poloidal field coils and some additional trim coils. The plasma is designed to be quasiaxisymmetric to give good transport and is stable to kink modes at $\beta \sim 4\%$, but is marginally unstable to infinite-n ballooning modes. The rotational-transform profile has $\iota \sim 0.4$ on axis, maximum $\iota \sim 0.66$ near the edge, and $\iota \sim 0.65$ at the edge: including the low-order resonances $\iota = 3/7$, $3/6$, and $3/5$. Note that the shear vanishes near the $\iota = 6/9$ resonance.

Considering a candidate coil set and selecting the $(n, m) = (3, 6), (3, 5)$ islands to be suppressed, subject to the constraint that the minimum coil curvatures, the coil-coil separation, and the kink stability be preserved

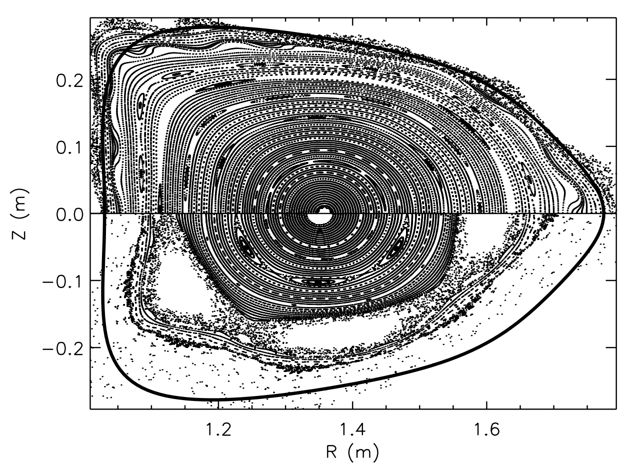

FIG. 1. Poincaré plot of the converged healed coil-plasma field (upper) and for the original, unhealed coils after 180 standard PIES iterations (lower) for the NCSX candidate coil set M45. The VMEC initialization boundary is shown as the thick solid line.

(9 constraints), and allowing some $m = 3, 4, 5, 6, 7, 8$ modular coil harmonics to vary (36 independent variables), a healed coil-plasma state is achieved. The engineering measures are preserved and the plasma is stable with respect to kink modes. Also, the plasma retains quasiaxisymmetry and is stable to ballooning modes $n < 45$.

Several hundred iterations are required to approach convergence in both the plasma field and the coil geometry. To confirm convergence, several hundred additional standard PIES iterations are performed with the coil set unchanged. A Poincaré plot of the final field is shown on an up-down symmetric toroidal cross section in the upper half of Fig. 1. The island content in the healed configuration is negligible, though there is some resonant $m = 18$ deformation near the zero shear location and some high-order ($m = 10, 11, 12$, and 14) island chains. For comparison, a Poincaré plot of the unhealed configuration is shown after 180 standard PIES iterations in the lower half of Fig. 1. For the unhealed case, there is a large $m = 5$ island and the configuration deteriorates into large regions of chaos.

The maximum coil alteration is about 2 cm, which comfortably exceeds manufacturing tolerances, but is not so large that "healing" significantly impacts other design concerns, such as diagnostic access. The coil harmonics varied actually describe the toroidal variation of the modular coils on a toroidal winding surface. The calculation shown used 63 radial surfaces, 12 poloidal, and 6 toroidal modes. Similar results have been obtained using up to 93 radial surfaces and 20 poloidal modes.

Comments.—The flux-surface quality of the "healed" equilibrium shows remarkable improvement compared to the unhealed configuration. The coils have been described with a filamentary model, and a finite thickness model of the healed coils shows further improvement; in particular, the $m = 18$ deformation and the high-order islands are reduced. To model a discharge evolution, we considered a sequence of equilibria with increasing plasma pressure. Though islands may reappear as the configuration departs from the healed configuration, the island content in each of the equilibria with the healed coils is much smaller than in the corresponding equilibrium with the original coils.

In principle, in the limit of suppressing additional islands, this approach can lead to nonaxisymmetric coil-plasma configurations with integrable magnetic fields. The procedure amounts to a stellarator design optimization routine that for the first time provides a mechanism for suppressing magnetic islands, while providing ideal stability and satisfying engineering constraints. In addition to the improvement in particle confinement associated with good-flux surfaces, the construction of integrable fields has implications for stellarator MHD stability calculations, which are usually based on equilibria artificially constrained to have nested flux surfaces. As the equilibria constructed using this method, and the method presented in [9], relax the unphysical imposition of nested surfaces, but nevertheless maintain integrability by careful design, stability studies based on these equilibria are expected to be more reliable.

We thank the NCSX design team, Long-Poe Ku, and Guo-Yong Fu for stability analysis, Raul Sanchez and Tony Cooper for use of the COBRA and TERPSICHORE codes, and Amitava Bhattacharjee and Chris Hegna for advice regarding the manuscript. This work was supported in part by U.S. Department of Energy Contract No. DE-AC02-76CH03073.

[1] A. H. Boozer, Phys. Plasmas **5**, 1647 (1998).
[2] J. R. Cary and R. G. Littlejohn, Ann. Phys. (N.Y.) **151**, 1 (1983); A. H. Boozer, Phys. Fluids **26**, 1288 (1983).
[3] A. H. Reiman, N. Pomphrey, and A. H. Boozer, Phys. Fluids B **1**, 555 (1989).
[4] A. H. Reiman and H. S. Greenside, Comput. Phys. Commun. **43**, 157 (1986).
[5] A. Bhattacharjee *et al.*, Phys. Plasmas **2**, 883 (1995).
[6] H. Grad, Phys. Fluids **10**, 137 (1967).
[7] J. R. Cary, Phys. Rev. Lett. **49**, 276 (1982); S. R. Hudson and R. L. Dewar, Phys. Lett. A **226**, 85 (1997).
[8] T. Hayashi *et al.*, Phys. Plasmas **1**, 3262 (1994); S. S. Lloyd *et al.*, J. Plasma Fusion Res. **1**, 484 (1998).
[9] S. R. Hudson, D. A. Monticello, and A. H. Reiman, Phys. Plasmas **8**, 3377 (2001).
[10] A. Reiman *et al.*, Phys. Plasmas **8**, 2083 (2001).
[11] S. P. Hirshman and O. Betancourt, J. Comput. Phys. **96**, 99 (1991).
[12] D. J. Strickler, L. A. Berry, and S. P. Hirshman, Fusion Sci. Technol. **41**, 107 (2001).
[13] A. H. Reiman and H. S. Greenside, J. Comput. Phys. **75**, 423 (1988).
[14] R. L. Dewar, S. R. Hudson, and P. Price, Phys. Lett. A **194**, 49 (1994).
[15] S. R. Hudson and R. L. Dewar, Phys. Plasmas **6**, 1532 (1999).
[16] W. H. Press *et al.*, *Numerical Recipes in Fortran 77: The Art of Scientific Computing* (Cambridge University, Cambridge, U.K., 1992), 2nd ed.
[17] W. A. Cooper, Plasma Phys. Controlled Fusion **34**, 1011 (1992).