Stability of Quantum States of Finite Macroscopic Systems against Classical Noises, Perturbations from Environments, and Local Measurements

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We study the stability of quantum states of macroscopic systems of finite volume V. By using both the locality and huge degrees of freedom, we show the following: (i) If square fluctuation of *every* additive operator is $O(V)$ or less for a pure state, then it is not fragile for *any* weak classical noises or weak perturbations from environments. (ii) If square fluctuation of *some* additive operator is $O(V^2)$ for a pure state, then it is fragile for *some* of these. (iii) If a state, pure or mixed, has the "cluster property," then it is stable against local measurements, and vice versa. Among many applications, we discuss the mechanism of symmetry-breaking in finite systems.

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The stability of quantum states of macroscopic systems, which are subject to weak classical noises (WCNs) or weak perturbations from environments (WPEs), have been studied in many fields of physics as the decoherence problem [1]. However, most previous studies assumed that the principal system was describable by a small number of collective coordinates. Although such models might be applicable to some systems, applicability to general systems is questionable. As a result of the use of such models, the results depended strongly on the choices of the coordinates and the form of the interaction $\hat{H}_{\rm int}$ between the principal system and a noise or an environment [1]. For example, a robust state for some $\hat{H}_{\rm int}$ can become a fragile state for another $\hat{H}_{\rm int}$. However, macroscopic physics and experiences strongly indicate that a more universal result should be drawn.

In this Letter, we study the stability of quantum states of finite macroscopic systems against WCNs and WPEs. We also propose a new criterion of stability; the stability against local measurements. We study these stabilities using a general model with a macroscopic number of degrees of freedom N. In addition to the fact that N is huge, we make full use of the locality [2,3]—"additive" observables must be the sum of local observables over a macroscopic region, the interaction $\hat{H}_{\rm int}$ must be local, and measurement must be local. By noticing these points, we derive general and universal results. Among many applications of the present theory, we discuss the mechanism of symmetry-breaking in finite systems.

Macroscopic quantum systems.—As usual, we are interested only in phenomena in some energy range ΔE and describe the system by an effective theory which correctly describes the system only in ΔE. For a given ΔE, let $\mathcal{M}$ be the number of *many-body* quantum states in that energy range. Then $N \sim \ln \mathcal{M}$ is the degrees of freedom of the effective theory. Note that N can become a small number even for a system of many degrees of freedom when, e.g., a non-negligible energy gap exists in ΔE, as in the cases of a heavy atom at a meV or lower energy range and SQUID systems at low temperatures. *We here exclude such systems,* because they are essentially systems of small degrees of freedom. Namely, we say that a system is macroscopic (for a given ΔE) only when its N is a macroscopic number. We further assume that the system extends homogeneously [4] over a volume V and that boundary effects are negligible. Since ΔE sets a minimum length scale ℓ, $V \sim N\ell^d$ in d dimension. We therefore say that V is also macroscopic. We study the stability of states of such a macroscopic system, when it is subject to WCNs, WPEs, and local measurements. The Hamiltonian $\hat{H}$ of the system can be a general one which has only short-range interactions.

Measures of the correlations between distant points.—As we show later, correlations between distant points are important. As a measure of the correlations, we first consider the cluster property. In *infinite* systems, a quantum state is said to have the cluster property if $\langle \delta\hat{a}(x)\delta\hat{b}(y)\rangle \to 0$ as $|x-y| \to \infty$ for any local operators $\hat{a}(x)$ and $\hat{b}(y)$ at x and y, respectively, where $\delta\hat{a}(x) \equiv \hat{a}(x) - \langle\hat{a}(x)\rangle$ and $\delta\hat{b}(y) \equiv \hat{b}(y) - \langle\hat{b}(y)\rangle$ [6]. Here, by a *local operator at x* we mean a finite-order polynomial of field operators and their finite-order derivatives at position x [3]. We generalize the concept of the cluster property to the case of finite systems as follows [9]. For a small positive number ϵ, we define a region $\Omega(\epsilon, x)$ by its complement $\Omega(\epsilon, x)^c$, which is the region of y in which

$$|\langle \delta\hat{a}(x)\delta\hat{b}(y)\rangle| \le \epsilon\sqrt{\langle \delta\hat{a}^\dagger(x)\delta\hat{a}(x)\rangle\langle \delta\hat{b}^\dagger(y)\delta\hat{b}(y)\rangle} \quad (1)$$

for any local operators $\hat{a}(x)$ and $\hat{b}(y)$. Let $\Omega(\epsilon) \equiv \sup_x |\Omega(\epsilon, x)|$, where $|\Omega(\epsilon, x)|$ denotes the size of $\Omega(\epsilon, x)$. Intuitively, $\Omega(\epsilon)$ is the size of the region outside which correlations of any local operators become negligible. We consider a sequence of homogeneous [4] systems with various values of V and associated states, where the shapes of V's are similar to each other. We say that the states (for large V) of the sequence have the cluster property if $\Omega(\epsilon)$ for any $\epsilon > 0$ becomes independent of V for a sufficiently large V. This means that $\Omega(\epsilon) \ll V$ if one takes V large enough. Note that a small number of Bell pairs do not destroy the cluster property: the lack of the cluster property means a *macroscopic* entanglement. As a second measure, we consider fluctuations of additive quantities. A physical quantity A is additive if $A = A^{(1)} + A^{(2)}$ when we regard the system as a composite system of subsystems 1 and 2. Thermodynamics *assumes* that any states in a pure phase satisfies $\langle(\delta A)^2\rangle = o(V^2)$ for *every* additive quantity. In particular, if a state of a (quantum or classical) system satisfies $\langle(\delta A)^2\rangle \leq O(V)$ for *every* additive quantity, we call it a "normally fluctuating state" (NFS). In *finite* quantum systems, on the other hand, there exist pure states for which *some* of the additive operators have anomalously large fluctuations; $\langle(\delta \hat{A})^2\rangle = O(V^2)$. We call such a *pure* state an "anomalously fluctuating state" (AFS). The locality requires that additive operators of quantum systems must have the following form: $\hat{A} = \sum_{x\in V} \hat{a}(x)$, where $\hat{a}(x)$ denotes a local operator at x. It is easy to show that an AFS does not have the cluster property, hence is entangled macroscopically. For *infinite* quantum systems, there is a well-known theorem: *Any pure state has the cluster property* [2]. Therefore, AFSs converge (in the weak topology) into *mixed* states as $V \to \infty$, although they are pure states in finite systems [10]. Since AFSs are such unusual states, they are expected to be unstable in some sense. We now clarify in what sense, how, and why unstable.

Fragility.—We say a quantum state is "fragile" if its decoherence rate Γ, Eq. (3), behaves as $\Gamma \sim KV^{1+\delta}$, where K is a function of microscopic parameters, and δ is a positive constant. To understand the meaning of the fragility, consider first the nonfragile case where $\delta = 0$. In this case, Γ/V is independent of V. This is a normal situation in the sense that the total decoherence rate Γ is basically the sum of *local* decoherence rates, which are determined only by microscopic parameters. On the other hand, the case where $\delta > 0$ is an anomalous situation in which $\Gamma/V \sim KV^{\delta}$. Note that this can be very large even when K is small, because, by definition, a macroscopic volume is huge. This means that *a fragile quantum state decoheres due to a noise or environment at an anomalously great rate,* even when the coupling constant between the system and the noise or environment is small.

Fragility in WCN.—The point of the present theory is the locality [2]. For the Hamiltonian $\hat{H}_{\rm int}$ of the interaction with a classical noise, the locality requires that it should be the sum of local interactions [11];

$$\hat{H}_{\rm int} = \lambda \sum_{x\in V} f(x,t)\hat{a}(x). \tag{2}$$

Here, λ is a small positive constant, $f(x,t)$ is a random classical noise field with vanishing average $\overline{f(x,t)} = 0$, and $\hat{a}(x)$ is a local operator at x. We assume that $\overline{f(x,t)f(x',t')}$ depends only on $x - x'$ and $t - t'$ and that its correlation time $\tau_{\rm c} \ll 1/\Gamma$ [12]. We denote the spectral intensity of f by $g(k,\omega)$ [13], which is positive by definition. A pure state $|\Psi\rangle$ at $t = 0$ evolves for $t > 0$ by the total Hamiltonian $\hat{H} + \hat{H}_{\rm int}$, and the density operator is given by $\hat{\rho}(t) \equiv \overline{|\Psi(t)\rangle\langle\Psi(t)|}$. Since we are interested in the dependence of Γ on the initial state, we study an early time stage $\tau_{\rm c} \ll t \ll 1/\Gamma$ and define Γ as the increase rate of the α entropy of $\alpha = 2$ in this time region;

$$\Gamma \equiv -\frac{1}{2}\frac{d}{dt}\ln{\rm Tr}[\hat{\rho}(t)^2]|_{\tau_{\rm c}\ll t\ll 1/\Gamma}. \tag{3}$$

Even when $\hat{H}_{\rm int} = 0$, $|\Psi\rangle$ generally evolves by $\hat{H}$. Since we are interested in the instability induced by $\hat{H}_{\rm int}$, we consider states which do not evolve by $\hat{H}$ in this time region, i.e., $\exp(-i\hat{H}t)|\Psi\rangle \simeq \exp(-i\langle\hat{H}\rangle t)|\Psi\rangle$ for such t. (However, see [14].) Moreover, since we are interested in the case of weak noise, we evaluate Γ to $O(\lambda^2)$. By dropping nondissipative contributions from $\hat{H}_{\rm int}$, because they can be absorbed in $\hat{H}$ as renormalization terms, we find $\Gamma \simeq \lambda^2 \sum_{k,n} g(k, \langle\hat{H}\rangle - \omega_n)|\langle n|\delta\hat{A}_k|\Psi\rangle|^2$. Here, $|n\rangle$ is an eigenstate of $\hat{H}$, with eigenenergy ω_n (which may be degenerate), and $\delta\hat{A}_k \equiv \hat{A}_k - \langle\Psi|\hat{A}_k|\Psi\rangle$, where $\hat{A}_k \equiv \sum_{x\in V}\hat{a}(x)e^{-ikx}$. If we put $\sum_n g(k, \langle\hat{H}\rangle - \omega_n) \times |\langle n|\delta\hat{A}_k|\Psi\rangle|^2 \equiv g(k)\sum_n |\langle n|\delta\hat{A}_k|\Psi\rangle|^2$, then $g(k)$ may be interpreted as a typical (average) value of $g(k, \langle\hat{H}\rangle - \omega_n)$ for relevant n's, because the two factors in the n summation are both positive. This interpretation would be good at least for the V dependence, which is our primary interest. We then obtain the simple formula

$$\Gamma \simeq \lambda^2 \sum_k g(k)\langle\Psi|\delta\hat{A}_k^\dagger\delta\hat{A}_k|\Psi\rangle. \tag{4}$$

Note that $\hat{A}_k$ is an additive operator because $\hat{a}(x)e^{-ikx}$ is a local operator. When $\hat{a}(x)$ is a spin operator, e.g., $\hat{A}_k$ for $k = \pi/\ell$ is the staggered magnetization.

When $|\Psi\rangle$ is an NFS, $\langle\Psi|\delta\hat{A}_k^\dagger\delta\hat{A}_k|\Psi\rangle \leq O(V)$ for any $\hat{A}_k$, hence $\Gamma \lesssim \lambda^2 O(V)\sum_k g(k)$. Since $\sum_k g(k, \omega_n - \omega_{n'}) = \int \overline{f(x,t)f(x,0)}e^{i(\omega_n-\omega_{n'})t}dt$ does not depend on V, neither does $\sum_k g(k)$. We thus find that *NFSs are not fragile in any WCN.* When $|\Psi\rangle$ is an AFS, on the other hand, $\langle\Psi|\delta\hat{A}_k^\dagger\delta\hat{A}_k|\Psi\rangle = O(V^2)$ for some $\hat{A}_k$, i.e., for some $\hat{a}(x)$ and some $k = k_0$. Hence, if $\hat{H}_{\rm int}$ has a term that is composed of such $\hat{a}(x)$'s, then

$$\Gamma \simeq \lambda^2 O(V^2)g(k_0) + \lambda^2 O(V)\sum_{k\neq k_0} g(k), \tag{5}$$

and the AFS becomes fragile if $g(k_0) = O(V^{-1+\delta})$, where $\delta > 0$. Therefore, *an AFS is fragile in some WCN.*

Fragility in WPE.—We obtain similar results for WPEs. From the locality, the interaction with an environment E should be the sum of local interactions; $\hat{H}_{\rm int} = \lambda\sum_{x\in V}\hat{f}(x)\hat{a}(x)$ [11]. Here, $\hat{f}(x)$ and $\hat{a}(x)$ are local operators at position x of E and the principal system, respectively. For $\langle\hat{f}(x,t)\rangle_E$ and $\langle\hat{f}(x,t)\hat{f}(x',t')\rangle_E$ (in the interaction picture), where $\langle\cdots\rangle_E$ denotes the expectation value for the state $\hat{\rho}_E$ of E, we assume the same properties as $\overline{f(x,t)}$ and $\overline{f(x,t)f(x',t')}$, respectively, of the WCN. The total Hamiltonian is $\hat{H} + \hat{H}_{\rm int} + \hat{H}_E$, where $\hat{H}_E$ is the

Hamiltonian of E. Taking the initial state $\hat{\rho}_{\rm total}(0)$ as the product state $|\Psi\rangle\langle\Psi| \otimes \hat{\rho}_E$, we evaluate the reduced density operator $\hat{\rho}(t) \equiv {\rm Tr}_E[\hat{\rho}_{\rm total}(t)]$. We then obtain the same result (4), where $g(k)$ is now a typical value of the spectral intensity derived from $\langle \hat{f}(x,t)\hat{f}(x',t')\rangle_E$. Therefore, *NFSs are not fragile under any WPE,* while *AFSs are fragile under some WPE* [16].

Summary of fragility.—We have shown that NFSs are not fragile in any WCNs or WPEs. This should be contrasted with the results of most previous works, according to which a state could be either fragile or robust depending on the form of $\hat{H}_{\rm int}$ [1]. Note that our results concern an *approximate* stability (i.e., nonfragility) against *all* possible WCNs or WPEs and $\hat{H}_{\rm int}$'s, whereas most previous works studied the *exact* stability against *particular* ones. We think that the former is more important in macroscopic systems because many types of WCNs or WPEs and $\hat{H}_{\rm int}$'s would coexist in real systems, and the exact stability against some of them could not exclude fragility to another. Regarding AFSs, on the other hand, our results show only that they are fragile in *some* WCN or WPE. In other words, for any AFS it is always possible to *construct* a noise (or an environment) and a weak local interaction with it in such a way that the AFS becomes fragile. These results do *not* guarantee the existence of the relevant noise (or an environment) and the relevant interaction *in real physical systems.* Since there is no theory that is general enough on WCNs or WPEs at present, we cannot draw a definite conclusion on whether AFSs are *always* fragile in real physical systems. It rather seems that, as we discuss later, there may be some cases where some AFSs are nonfragile, in contradiction to naive expectations. This motivates us to explore the following new criterion of stability.

Stability against local measurements.—Suppose that one performs an ideal (von Neumann) measurement of a local observable $\hat{a}(x)$ at $t = t_a$ for a state $\hat{\rho}$ (pure or mixed) of a macroscopic system and obtains a value a with a finite probability $P(a) \neq 0$. Subsequently, one measures another local observable $\hat{b}(y)$ at a later time t_b [17] and obtains a value b. Let $P(b;a)$ be the probability that b is obtained at t_b under the condition that a was obtained at t_a. On the other hand, one can measure $\hat{b}(y)$ at $t = t_b$ without performing the measurement of $\hat{a}(x)$ at t_a. Let $P(b)$ be the probability distribution of b in this case. We say $\hat{\rho}$ is "stable against local measurements" if for any $\varepsilon > 0$

$$|P(b;a) - P(b)| \leq \varepsilon \quad \text{for sufficiently large } |x-y|, \tag{6}$$

for *any* local operators $\hat{a}(x)$ and $\hat{b}(y)$ and their eigenvalues a and b such that $P(a) \geq \varepsilon$ [18]. For the simplest case $t_b \to t_a$, we obtain the simple theorem: *If $\hat{\rho}$ is stable against local measurements, then it has the cluster property, and that any state which has the cluster property is stable against local measurements.* It follows, e.g., that any AFS is unstable against local measurements.

To prove this theorem, we use the spectral decomposition; $\hat{a}(x) = \sum_a a\hat{\mathcal{P}}_a(x)$ and similarly for $\hat{b}(y)$. Here, $\hat{\mathcal{P}}_a(x)$ denotes the projection operator corresponding to an eigenvalue a of $\hat{a}(x)$. Since we are considering an effective theory in a finite energy range, we assume that ultraviolet divergences are absent: e.g., for any positive integer m, $\langle \hat{a}(x)^m\rangle =$ finite for any local operator $\hat{a}(x)$. For $t_b \to t_a$, both $|P(b;a) - P(b)| \leq \varepsilon$ and $|P(a;b) - P(a)| \leq \varepsilon$ are satisfied if $\hat{\rho}$ is stable against local measurements. Expressing the probabilities by the projection operators, we obtain $|{\rm Tr}[\hat{\rho}\hat{\mathcal{P}}_a(x)\hat{\mathcal{P}}_b(y)] - {\rm Tr}[\hat{\rho}\hat{\mathcal{P}}_a(x)] \times {\rm Tr}[\hat{\rho}\hat{\mathcal{P}}_b(y)]| \leq \varepsilon \min[P(a), P(b)]$ for $P(a), P(b) \geq \varepsilon$. Multiplying this equation by $|ab|$, and summing over a and b such that $P(a), P(b) \geq \varepsilon$, we can show the cluster property. To prove the inverse, we take $\hat{a}(x) = \hat{\mathcal{P}}_a(x)$, $\hat{b}(y) = \hat{\mathcal{P}}_b(y)$. Then, from (1), $|\langle \delta\hat{\mathcal{P}}_a(x)\delta\hat{\mathcal{P}}_b(y)\rangle| \leq \epsilon\sqrt{P(a)[1-P(a)]P(b)[1-P(b)]}$ for sufficiently large $|x-y|$. Dividing this by $P(a)$ yields $|P(b;a) - P(b)| \leq \sqrt{\epsilon}$ for $P(a) \geq \epsilon$ [hence, also for $P(a) \geq \sqrt{\epsilon}$]. We thus obtain the stability against local measurements.

Applications.—The above results have many applications, including quantum computers with many qubits [5], and nonequilibrium statistical physics. We here discuss the mechanism of symmetry breaking (SB) in *finite* systems, which has been a long-standing question for the following reasons. Consider a finite system that will exhibit a SB if V goes to infinity. Let $|\Psi\rangle_V$ be a state that approaches, as $V \to \infty$, a SB vacuum $|\Psi\rangle_\infty$ of the infinite system. We call $|\Psi\rangle_V$ for large V a pure-phase vacuum. It has a macroscopic value $\langle \hat{M}\rangle = O(V)$ of an additive order parameter $\hat{M}$. In a mean-field approximation, pure-phase vacua have the lowest energy. However, it is *always* possible (see the example below) to construct a pure state(s) that does not break the symmetry, $\langle \hat{M}\rangle = 0$, and has an equal or *lower* energy than pure-phase vacua [7,15]. Although such states cannot be pure in infinite systems, they can be pure in finite systems [2,7,8,15]. When $[\hat{H}, \hat{M}] \neq 0$, in particular, the exact lowest-energy state is generally such a symmetric ground state [7,15]. To lower the energy of a pure-phase vacuum, a SB field is necessary. However, an appropriate SB field would not always exist in laboratories. For example, the SB field for antiferromagnets is a static staggered magnetic field, which alters its direction at the period exactly twice the lattice constant. It seems quite unlikely that such a field would always exist in laboratories.

Our results suggest the following new mechanisms of SB in finite systems. From the well-known theorem mentioned earlier, $|\Psi\rangle_\infty$ has the cluster property. Since $|\Psi\rangle_V$ approaches $|\Psi\rangle_\infty$, it also has the cluster property for large V. Hence, pure-phase vacua are not AFSs. On the other hand, $\langle(\delta\hat{M})^2\rangle = O(V^2)$ for the symmetric ground state because it is composed primarily of a superposition of pure-phase vacua with different values of $\langle \hat{M}\rangle$ [7,15]. Namely, the symmetric ground state is an AFS and, thus, is fragile in some WCN or WPE. Therefore, we expect that a pure-phase vacuum would be realized much more easily than the symmetric ground state. This mechanism may be called "environment-induced symmetry breaking," a special case of which was discussed for interacting many-bosons [8]. For general systems, however, there is one delicate point: $g(k_0)$ of the relevant WCN or WPE might be $O(1/V)$ in some of real systems [19]. Then, Eq. (5) yields $\Gamma = O(V)$, and the symmetric ground state becomes nonfragile. In such a case, we must consider the stability against local measurements: Even when the symmetric ground state is somehow realized at some time, it is changed into another state when one measures a relevant observable that is localized within only a tiny part of the system. Such drastic changes continue by repeating measurements of relevant observables, until the state becomes a pure-phase vacuum and the symmetry is broken. This mechanism may be called "measurement-induced symmetry breaking." We conjecture (and confirm in several examples) that the number of local measurements necessary for reducing an AFS to an NFS would be much less than N. For example, if we regard the spins of the antiferromagnetic Ising model as quantum spins, the pure-phase vacua are the Néel states, $|\Psi_+\rangle \equiv |\uparrow\downarrow\uparrow \cdots \downarrow\rangle$ and $|\Psi_-\rangle \equiv |\downarrow\uparrow\downarrow \cdots \uparrow\rangle$, for which the staggered magnetization $\hat{M}_\pi \equiv \sum_x e^{i\pi x/\ell}\hat{\sigma}_z(x)$ is the order parameter; $\langle\Psi_\pm|\hat{M}_\pi|\Psi_\pm\rangle = \pm V$. On the other hand, $|\Phi\rangle \equiv (|\Psi_+\rangle + |\Psi_-\rangle)/\sqrt{2}$ is a symmetric ground state, degenerating with $|\Psi_\pm\rangle$. It is an AFS because $\langle\Phi|\delta\hat{M}_\pi^2|\Phi\rangle = V^2$. According to our results, $|\Psi_\pm\rangle$ are stable, whereas $|\Phi\rangle$ are unstable, against local measurements. Therefore, if the initial state is $|\Phi\rangle$, it is drastically altered by a measurement of only a tiny part of the system. For example, by measurement of $\hat{\sigma}_z$ of the first spin, $|\Phi\rangle$ reduces to either $|\Psi_+\rangle$ when $\sigma_z = +1$ or $|\Psi_-\rangle$ when $\sigma_z = -1$. Namely, the symmetric ground state turns into a pure-phase vacuum after the local measurement, and the symmetry is then broken. After that, the state alters only slightly by subsequent local measurements because $|\Psi_\pm\rangle$ are stable against local measurements.

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[1] See, e.g., *Quantum Coherence and Decoherence*, edited by Y. A. Ono and K. Fujikawa (North-Holland, Amsterdam, 1999).

[2] R. Haag, *Local Quantum Physics* (Springer, Berlin, 1992).

[3] To express and utilize the locality of the theory manifestly, we use a field theory throughout this Letter.

[4] The homogeneity can be a generalized one: Our theory is applicable to, e.g., quantum computers [5], in which states for some N are similar to states for another N because they are generated by the same algorithm.

[5] A. Shimizu *et al.* (unpublished).

[6] The cluster property should not be confused with the absence of the long-range order. In fact, symmetry-breaking vacua have *both* the long-range order and the cluster property [2,7,8]. Note also that domain walls do *not* destroy the cluster property.

[7] P. Horsch and W. von der Linden, Z. Phys. B **72**, 181 (1988); T. Koma and H. Tasaki, J. Stat. Phys. **76**, 745 (1994).

[8] A. Shimizu and T. Miyadera, Phys. Rev. Lett. **85**, 688 (2000); J. Phys. Soc. Jpn. **71**, 56 (2002).

[9] T. Miyadera, Ph.D. thesis, University of Tokyo, 2001.

[10] In the Ising model discussed below, $|\Psi_+\rangle$ and $|\Psi_-\rangle$ become in the limit of $V \to \infty$ vectors of inequivalent Hilbert spaces $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively. Since there is no interference between them, $|\Phi\rangle \in \mathcal{H}_+ \oplus \mathcal{H}_-$ in this limit is equivalent to a classical mixture of $|\Psi_+\rangle$ and $|\Psi_-\rangle$.

[11] Although we describe here the case where f or $\hat{f}$ and $\hat{a}$ are real or Hermitian, the generalization is straightforward.

[12] For a given V, this is satisfied for sufficiently small λ, e.g., $1/\Gamma \sim 1$ ms and 1 ns, respectively, for a NFS and an AFS with $N = 10^6$, whereas $\tau_c \sim 1$ ps. Although Γ varies with λ, we stress that Γ for AFSs can be much larger than Γ for NFSs by many orders of magnitude for any value of λ, as shown below.

[13] Since $f(x, t)$ for $x \notin V$ is irrelevant, the spatial Fourier transform is taken over V to define $g(k, \omega)$; hence, k takes discrete values with separation $\sim V^{-1/d}$.

[14] When this condition is violated around $t \sim t_1$ such that $\tau_c \ll t_1 \ll 1/\Gamma$, we can apply formula (4) again now to the evolved state $e^{-i\hat{H}t_1}|\Psi\rangle$, because the decoherence until t_1 does not affect Γ after t_1 to $O(\lambda^2)$. We may repeat this for intervals $[t_1, t_2), [t_2, t_3), \ldots, [t_{M-1}, t_M)$ until some $t_M \ll 1/\Gamma$. For a rotating ferromagnet and a condensed state of bosons [15], the only quick evolution induced by $\hat{H}$ is the rotation of the direction and phase, respectively, of the order parameter. In such cases, Γ in every interval is of the same order, and formula (4) gives the correct order of magnitude in the entire region of $\tau_c \ll t \le t_M$, although $e^{-i\hat{H}t}|\Psi\rangle \simeq e^{-i\langle\hat{H}\rangle t}|\Psi\rangle$ is not satisfied for $t > t_1$.

[15] A. Shimizu and T. Miyadera, Phys. Rev. E **64**, 056121 (2001).

[16] A special case of this result was obtained by G. M. Palma *et al.*, Proc. R. Soc. London A **452**, 567 (1996).

[17] (x, t_a) and (y, t_b) are separated either timelike or spacelike.

[18] The case of continuous eigenvalues can be discussed in a similar manner.

[19] For example, electromagnetic noises at 4 K contribute only to the $k = 0$ component of $g(k)$ if the diameter of the system is less than 1 cm [13]. Since $\sum_k g(k) = O(V^0)$, $g(0) = O(V^0)$, whereas $g(k) = O(1/V)$ for $k \gg 1$ cm^{-1}.