# New Dynamical Mean-Field Dynamo Theory and Closure Approach

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We develop a new nonlinear mean field dynamo theory that couples field growth to the time evolution of the magnetic helicity and the turbulent electromotive force, $\overline{\mathcal{E}}$. We show that the difference between kinetic and current helicities emerges naturally as the growth driver when the time derivative of $\overline{\mathcal{E}}$ is coupled into the theory. The solutions predict significant field growth in a kinematic phase and a saturation rate/strength that is magnetic Reynolds number dependent/independent in agreement with numerical simulations. The amplitude of early time oscillations provides a diagnostic for the closure.

Mean field dynamo (MFD) theory has been a useful framework for modeling the *in situ* origin of large-scale magnetic field growth in planets, stars, and galaxies [1–4], and has also been invoked to explain the sustenance of fields in fusion devices [5,6]. However, whether the backreaction of the magnetic field itself prematurely quenches the MFD has been debated [7–24]. Recent progress has emerged from incorporating magnetic helicity evolution into the theory.

To make explicit the problem to be solved, we first average the magnetic induction equation to obtain the basic MFD equation [1,3]:

$$\partial_t\overline{\mathbf{B}} = \nabla\times\overline{\mathcal{E}} + \nabla\times(\overline{\mathbf{V}}\times\overline{\mathbf{B}}) - \lambda\nabla^2\overline{\mathbf{B}}, \qquad (1)$$

where $\overline{\mathbf{B}}$ is the mean (large-scale) magnetic field in Alfvén speed units, $\lambda = (\eta c^2)/(4\pi)$ is the magnetic diffusivity in terms of the resistivity $\eta$, $\overline{\mathbf{V}}$ is the mean velocity which we set equal to 0, and $\overline{\mathcal{E}} = \langle\mathbf{v}\times\mathbf{b}\rangle$ is the turbulent electromotive force, a correlation between fluctuating velocity $\mathbf{v}$ and magnetic field $\mathbf{b}$ in Alfvén units. Textbook treatments [1,3] invoke $\overline{\mathcal{E}} = \alpha\overline{\mathbf{B}} - \beta\nabla\times\overline{\mathbf{B}}$, where $\alpha$ and $\beta$ are pseudoscalar and scalar correlations of turbulent quantities, respectively. In the kinematic theory [1] $\alpha = -(\tau_c/3)\langle\mathbf{v}\cdot\nabla\times\mathbf{v}\rangle$, where $\tau_c$ is a correlation time, and (1) is solved with $\alpha$ and $\beta$ as input parameters.

But the kinematic theory is incomplete. In a study of helical MHD turbulence, Ref. [18] derived approximate evolution equations for the spectra of kinetic energy, magnetic energy, kinetic helicity, and magnetic helicity ($\equiv \langle\mathbf{A}\cdot\nabla\times\mathbf{A}\rangle$, where $\mathbf{B} = \nabla\times\mathbf{A}$). These calculations suggested that $\alpha \simeq (\tau_c/3)\langle\mathbf{b}\cdot\nabla\times\mathbf{b} - \mathbf{v}\cdot\nabla\times\mathbf{v}\rangle$, the residual helicity, where $\langle\ \rangle$ indicates spatial or ensemble average. This form has been employed in attempts to understand nonlinear dynamo quenching by coupling magnetic helicity conservation into the dynamo through the $\langle\mathbf{b}\cdot\nabla\times\mathbf{b}\rangle$ term [4,19–23]. Although these studies [e.g., [4,19,20]] wrote down an equation for the time evolution of $\alpha$, they derived quenching formulas for $\alpha$

only for the steady state. Only after a coupled nonlinear system of time-dependent large- and small-scale magnetic helicity equations were solved [15] was it apparent that a dynamical quenching model based on residual helicity reveals both a kinematic growth phase and an asymptotic resistively limited phase as seen in numerical experiments [12]. The dynamical approach has also been applied to dynamos with shear [16].

But even in these dynamical approaches, the $\overline{\mathcal{E}}$ was assumed to be proportional to the residual helicity. Here we show that the required residual helicity emerges not from $\overline{\mathcal{E}}$, but from $\partial_t\overline{\mathcal{E}}$, and that including the $\partial_t\overline{\mathcal{E}}$ equation in addition to the MFD and total magnetic helicity equations is essential for a complete MFD theory. We first derive $\partial_t\overline{\mathcal{E}}$ and then derive the triplet of equations to be solved for the simple shear-free helical dynamo whose solutions can be compared with existing numerical simulations. We discuss these solutions, physical implications, and the relation to previous work.

*Deriving $\partial_t\overline{\mathcal{E}}$.*—A two-scale nonlinear quenching approach invoking the residual helicity in $\overline{\mathcal{E}}$ [15] captures the nonlinear dynamo saturation seen in simulations [12], but the derivation of the appropriate $\overline{\mathcal{E}}$ has been elusive. To couple $\langle\mathbf{b}\cdot\nabla\times\mathbf{b}\rangle$ of $\overline{\mathcal{E}}$ to the magnetic helicity conservation equation, the full small-scale field $\mathbf{b}$ must enter this correlation not a low order approximation. The essence of the puzzle [17] is that

$$\overline{\mathcal{E}}(t) = \langle\mathbf{v}(t)\times\mathbf{b}(t)\rangle = \int_0^t\langle\mathbf{v}(t)\times\partial_{t'}\mathbf{b}(t')\rangle dt'$$
$$= -\int_0^t\langle\mathbf{b}(t)\times\partial_{t'}\mathbf{v}(t')\rangle dt', \qquad (2)$$

assuming that $b(0) = 0$ and that $t \gg 0$ so that $\langle\mathbf{v}(0)\times\mathbf{b}(t)\rangle = 0$. The last two terms are both exact expressions for $\overline{\mathcal{E}}$; however, neither leads naturally the residual helicity entering $\alpha$: Upon using the induction equation for $\mathbf{b}$, the second term on the right leads to a term $\propto \int\langle\mathbf{v}(t)\nabla\times\mathbf{v}(t)\rangle dt'$. Using the Navier-Stokes equation for $\mathbf{v}$, the last term contributes a term $\propto \int\langle\mathbf{b}(t)\cdot\nabla\times\mathbf{b}(t)\rangle dt'$. One emerges with a *choice* rather than the *difference* between

the two helicities. So how does the residual helicity emerge?

Rather than impose the form of the $\overline{\mathcal{E}}$, we solve for it dynamically using

$$\partial_t \overline{\mathcal{E}} = \langle \partial_t \mathbf{v} \times \mathbf{b} \rangle + \langle \mathbf{v} \times \partial_t \mathbf{b} \rangle. \tag{3}$$

To proceed, we need equations for $\partial_t \mathbf{b}$ and $\partial_t \mathbf{v}$. Assuming $\nabla \cdot \mathbf{v} = 0$, we have

$$\partial_t \mathbf{b} = \overline{\mathbf{B}} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \overline{\mathbf{B}} + \nabla \times (\mathbf{v} \times \mathbf{b}) - \nabla \times \langle \mathbf{v} \times \mathbf{b} \rangle + \lambda \nabla^2 \mathbf{b}, \tag{4}$$

and

$$\partial_t v_q = P_{qi}(\overline{\mathbf{B}} \cdot \nabla b_i + \mathbf{b} \cdot \nabla \overline{B}_i - \mathbf{v} \cdot \nabla v_i + \langle \mathbf{v} \cdot \nabla v_i \rangle + \mathbf{b} \cdot \nabla b_i - \langle \mathbf{b} \cdot \nabla b_i \rangle) + \nu \nabla^2 v_q + f_q, \tag{5}$$

where $\mathbf{f}$ is a divergence-free forcing function uncorrelated with $\mathbf{b}$, $\nu$ is the viscosity, and $P_{qi} \equiv (\delta_{qi} - \nabla^{-2} \nabla_q \nabla_i)$ is the projection operator that arises after taking the divergence of the incompressible Navier-Stokes equation to eliminate the total fluctuating pressure (magnetic + thermal). Using Reynolds rules [25] to interchange brackets with time and spatial derivatives, the fifth term of (4) and the fourth and sixth terms in the parentheses of (5) do not contribute when put into the averages, so we ignore them.

The contribution to $\partial_t \overline{\mathcal{E}}$ from the third term in (3) can be derived by direct use of (4) in configuration space. We assume isotropy of the resulting velocity and magnetic field correlations for terms linear in $\overline{\mathbf{B}}$. We also retain the triple correlations. The contribution to $\partial_t \overline{\mathcal{E}}$ from the second term in (3) also contributes terms linear in $\overline{\mathbf{B}}$, and triple correlations. Here the terms linear in $\overline{\mathbf{B}}$ are best derived in Fourier space. For this, we follow the technique in the appendix of Ref. [22], which invokes the Fourier transform of the terms linear in $\overline{\mathbf{B}}$ contributing to $\langle \partial_t \mathbf{v} \times \mathbf{b} \rangle$, supplemented by a linear expansion of the projection operator in $k_1 \ll k_2$, where $k_1$ is the characteristic wave number of the bracketed or mean quantities, and $k_2$ is the characteristic wave number of the fluctuating quantities $\mathbf{b}$ and $\mathbf{v}$. Collecting all surviving terms, we then have for (3)

$$\partial_t \overline{\mathcal{E}} = \tfrac{1}{3}(\langle \mathbf{b} \cdot \nabla \times \mathbf{b} \rangle - \langle \mathbf{v} \cdot \nabla \times \mathbf{v} \rangle) \overline{\mathbf{B}} - \tfrac{1}{3} \langle \mathbf{v}^2 \rangle \nabla \times \overline{\mathbf{B}} + \nu \langle \nabla^2 \mathbf{v} \times \mathbf{b} \rangle + \lambda \langle \mathbf{v} \times \nabla^2 \mathbf{b} \rangle + \mathbf{T}^V + \mathbf{T}^M, \tag{6}$$

where $\mathbf{T}^M = \langle \mathbf{v} \times \nabla \times (\mathbf{v} \times \mathbf{b}) \rangle$ and $T_j^V = \langle \epsilon_{jqn} P_{qi}(\mathbf{b} \cdot \nabla b_i - \mathbf{v} \cdot \nabla v_i) b_n \rangle$ are the triple correlations. Note that the third, fourth, sixth, and eighth terms in (6) come from the $\langle \mathbf{v} \times \partial_t \mathbf{b} \rangle$ term of (3), and the second, fifth, and seventh terms come from the $\langle \partial_t \mathbf{v} \times \mathbf{b} \rangle$ term of (3).

We are primarily interested in the component of $\overline{\mathcal{E}}$ parallel to $\overline{\mathbf{B}}$. For this we have

$$\partial_t \overline{\mathcal{E}}_{||} = (\langle \partial_t \mathbf{v} \times \mathbf{b} \rangle + \langle \mathbf{v} \times \partial_t \mathbf{b} \rangle) \cdot \overline{\mathbf{B}}/|\overline{\mathbf{B}}| + \langle \mathbf{v} \times \mathbf{b} \rangle \times \partial_t (\overline{\mathbf{B}}/|\overline{\mathbf{B}}|). \tag{7}$$

Substituting (6) into (7) gives

$$\partial_t \overline{\mathcal{E}}_{||} = \tilde{\alpha} \overline{\mathbf{B}}^2/|\overline{\mathbf{B}}| - \tilde{\beta} \overline{\mathbf{B}} \cdot \nabla \times \overline{\mathbf{B}}/|\overline{\mathbf{B}}| - \tilde{\zeta} \overline{\mathcal{E}}_{||}, \tag{8}$$

where $\tilde{\alpha} = (1/3)(\langle \mathbf{b} \cdot \nabla \times \mathbf{b} \rangle - \langle \mathbf{v} \cdot \nabla \times \mathbf{v} \rangle)$, $\tilde{\beta} = (1/3)\langle \mathbf{v}^2 \rangle$, and $\tilde{\zeta}$ accounts for microphysical dissipation terms, the last term of (7), and any additional contribution arising from $\mathbf{T}^M + \mathbf{T}^V \neq 0$. Note that $\tilde{\alpha}$ and $\tilde{\beta}$ appear similar to the usual $\alpha$ and $\beta$ dynamo coefficients in $\overline{\mathcal{E}}$, but they are fundamentally different because they are coefficients in $\partial_t \overline{\mathcal{E}}$ (and thus have different units) and do not involve $\tau_c$. Note also that if isotropy of like correlations were strongly violated, $\tilde{\alpha}$ and $\tilde{\beta}$ would be anisotropic tensors in analogy to tensor generalizations of $\alpha$ and $\beta$ [26,27]. We have not considered that here.

Following [15], we define large- and small-scale magnetic helicities as $H_1^M \equiv \langle \overline{\mathbf{A}} \cdot \overline{\mathbf{B}} \rangle_{\text{vol}}$ and $H_2^M \equiv \langle \mathbf{a} \cdot \mathbf{b} \rangle_{\text{vol}}$, where $\langle \rangle_{\text{vol}}$ indicates a global spatial average. Then $\langle \mathbf{b} \cdot \nabla \times \mathbf{b} \rangle_{\text{vol}} = k_2^2 H_2^M$ and $\langle \overline{\mathbf{B}} \cdot \nabla \times \overline{\mathbf{B}} \rangle_{\text{vol}} = k_1^2 H_1^M$. We define the small-scale kinetic helicity $H_2^V = \langle \mathbf{v} \cdot \nabla \times \mathbf{v} \rangle$. We assume $\overline{\mathbf{V}} = 0$, and a force-free large-scale field for which the $\partial_t H_1^M$ equation becomes degenerate [15] with that of $\partial_t \overline{\mathbf{B}}^2$. Then $|\overline{\mathbf{B}}| = k_1^{1/2}|H_1^{1/2}|$. We can thus rewrite (3) as

$$\partial_t \overline{\mathcal{E}}_{||} = k_1^{1/2}|H_1^M|^{1/2}(k_2^2 H_2^M - H_2^V)/3 - k_1^{3/2}(H_1^M/(|H_1^M|^{1/2}) \tilde{\chi} - \tilde{\zeta} \overline{\mathcal{E}}_{||}. \tag{9}$$

*Dynamo equations.*—We couple (9) to the equations for small- and large-scale magnetic helicity evolution for a dynamo in which the kinetic energy is externally forced and $\overline{\mathbf{V}} = 0$. We interpret $\overline{\mathbf{B}}$, $\overline{\mathbf{A}}$, and $\overline{\mathcal{E}}$ as the $k_1$ ($0 < k_1 < k_2$) component of $\mathbf{B}$, $\mathbf{A}$ and $(\mathbf{v} \times \mathbf{b})$ of a closed system to facilitate comparison with simulations of Ref. [12]. The total magnetic helicity, $H^M = \langle \mathbf{A} \cdot \mathbf{B} \rangle_{\text{vol}}$, then satisfies [1] $\partial_t \langle \mathbf{A} \cdot \mathbf{B} \rangle_{\text{vol}} = -2 \langle \mathbf{E} \cdot \mathbf{B} \rangle_{\text{vol}}$, where $\mathbf{E} = -\partial_t \mathbf{A} - \nabla \phi$, and $\phi$ is the scalar potential. The large- and small-scale integrated magnetic helicity equations are then [12,15,28]

$$\partial_t \langle \overline{\mathbf{A}} \cdot \overline{\mathbf{B}} \rangle_{\text{vol}} = 2 \langle \overline{\mathcal{E}} \cdot \overline{\mathbf{B}} \rangle_{\text{vol}} - 2\lambda \langle \overline{\mathbf{B}} \cdot \nabla \times \overline{\mathbf{B}} \rangle_{\text{vol}}, \tag{10}$$

and

$$\partial_t \langle \mathbf{a} \cdot \mathbf{b} \rangle_{\text{vol}} = -2 \langle \overline{\mathcal{E}} \cdot \overline{\mathbf{B}} \rangle_{\text{vol}} - 2\lambda \langle \mathbf{b} \cdot \nabla \times \mathbf{b} \rangle_{\text{vol}}. \tag{11}$$

When $\overline{\mathbf{B}}$ is force-free, the two-scale approximation allows us to write (10) and (11) as

$$\partial_t H_1^M = 2\overline{\mathcal{E}}_{||} k_1^{1/2}|H_1^M|^{1/2} - 2\lambda k_1^2 H_1^M, \tag{12}$$

and

$$\partial_t H_2^M = -2\overline{\mathcal{E}}_{||} k_1^{1/2}|H_1^M|^{1/2} - 2\lambda k_2^2 H_2^M. \tag{13}$$

We need to solve (12), (13), and (9) after converting them into dimensionless form. We define the dimensionless quantities $h_1 \equiv H_1^M(k_2/v_2^2)$ and $h_2 \equiv H_2^M(k_2/v_2^2)$, $R_M \equiv v_2/\lambda k_2$, $Pr_M \equiv \nu/\lambda$, $\tau \equiv t v_2 k_2$, $Q = -\overline{\mathcal{E}}_{||}/v_2^2$, $\chi = \tilde{\beta}/v_2^2$, $\zeta = \tilde{\zeta}/v_2 k_2 = (1 + Pr_M)/R_M$, and use $H_2^V = -k_2 v_2^2$. For (12), (13), and (9), respectively, this gives

$$\partial_\tau h_1 = -2Q h_1^{1/2}(k_1/k_2)^{1/2} - 2h_1 (k_1/k_2)^2/R_M, \tag{14}$$

$$\partial_\tau h_2 = 2Q h_1^{1/2}(k_1/k_2)^{1/2} - 2h_2/R_M, \tag{15}$$

and

$$\partial_\tau Q = -(k_1/k_2)^{1/2}h_1^{1/2}(1+h_2)/3 + (k_1/k_2)^{3/2}h_1^{1/2}\chi$$
$$- \zeta Q. \tag{16}$$

*Solutions.*—Since $H_2^V < 0$, then $H_1^M > 0$ and $H_2^M < 0$ for a growing solution. The solutions of (14)–(16) for two different $R_M$ are shown in Figs. 1 and 2 over different time ranges for both $\zeta = 2/R_M$ and $\zeta = 1$; we use $\tilde{\alpha} \propto \tilde{\beta}$ in Fig. 1, but the resulting solutions are only weakly sensitive to the form of $\tilde{\beta}$ as shown in Fig. 2. In Figs. 1 and 2, we also plotted the empirical fit formula to numerical simulations [12] [Eq. (54) of Ref. [12]] using our dimensionless parameters, to demonstrate the good agreement.

In Figs. 1 and 2, we also compare the triplet solution of $h_1$ with the doublet solution [15] that results from solving (14) and (15) but with *imposing* $\overline{\mathcal{E}} = \alpha\overline{\mathbf{B}} - \beta\nabla \times \overline{\mathbf{B}}$ such that $Q = Q_d \equiv -(k_1/k_2)^{1/2}h_1^{1/2}(1+h_2)\tau_c/3 + (k_1/k_2)^{3/2}h_1^{1/2}\chi\tau_c$, where the correlation time $\tau_c$ is a free parameter taken to be $\sim 1$. Note that the present triplet solution does not involve $\tau_c$ in the dynamo coefficients $\tilde{\alpha}$ and $\tilde{\beta}$. But a remarkable result emerges: Fig. 1 shows that the triplet solution matches the doublet solution at early times for $\zeta = 1$, which corresponds to a damping time $\zeta^{-1} \sim \tau_c$. This arises from a closure in which the triple correlations $\mathbf{T}^M$ and $\mathbf{T}^V$ lead to a damping with time constant $\sim\tau_c$, and the damping suppresses the oscillations. Figure 2 also shows that the $\zeta = 2/R_M$ and $\zeta = 1$ cases are indistinguishable at late times. We can also compare the kinematic regimes of the triplet and

doublet solutions. The rise to the first peak of $h_1$ in Fig. 1(a) is independent of $R_M$, and there the two $R_M$ triplet solutions overlap. This is the kinematic regime. The end of the doublet kinematic regime occurs at $h_1 \simeq 1$, as seen in Fig. 1. Again, the doublet and triplet match when $\zeta \sim 1$. In summary, *the doublet solution emerges as the limit of the triplet solution when the triple correlations act as a damping term*. This closure can be tested with future simulations.

The maximum kinematic growth rate for $h_1$ is a function of $\zeta$ because it occurs where $Q$ is a minimum. If we ignore resistive terms so that $h_1 = -h_2$, and assume that $\zeta \ll 1$ and $\chi = 1/3$, then Eq. (16) implies that the maximum growth rate occurs when $h_1 \simeq 1 - k_1/k_2$. From Fig. 1(a), the minimum of $Q$ during the first oscillation is $\sim -1/3$ (found to be independent of $k_1/k_2$). Setting $\partial_\tau h_1 \sim nh_1$, the maximum kinematic growth rate from (14) is then $n \sim (2/3)(k_1/k_2)^{1/2}(1 - k_1/k_2)^{-1/2} \sim 0.33$, for $k_2 = 5k_1$. However, when $\zeta = 1$, the minimum of $Q$ from (16) occurs where $Q = Q_d$. In this case, $n \sim (2/3)(k_1/k_2)(1 - k_1/k_2) \sim 0.11$ for $k_2 = 5k_1$. This demonstrates that the kinematic growth rate for $\zeta \ll 1$ is $\sim 3$ times that for $\zeta = 1$, and why the triplet kinematic growth rate with $\zeta = 1$ matches that of the doublet. Both results are seen in Fig. 1.

Inspection of (14)–(16), reveals why there are oscillations for a positive seed $h_1$ and $\zeta \ll 1$ [and independent of whether $\overline{\mathcal{E}}(t = 0) = 0$ or $\overline{\mathcal{E}}(t = 0) \neq 0$]. As long as
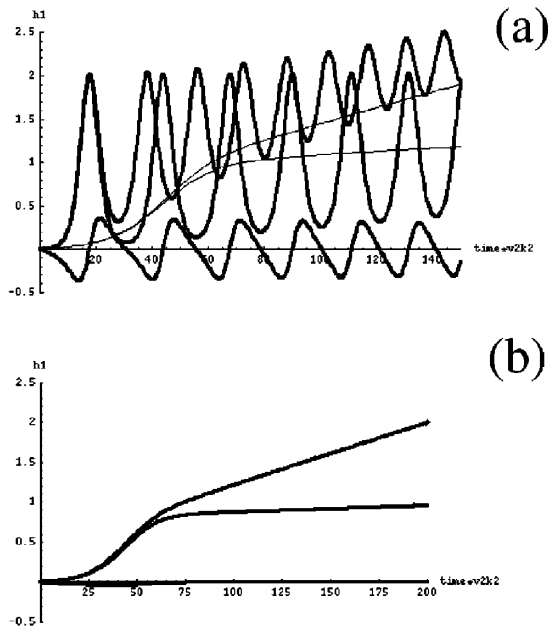
FIG. 1. (a) Plot for $k_1 = 1$, $k_2 = 5$ with $\zeta = 2/R_M$. The top and middle oscillating curves are for $h_1$ with $R_M = 200$ and $R_M = 1000$, and the bottom oscillating curve is $Q$ for $R_M = 1000$. The thin lines are the doublet solutions for $h_1$ from Ref. [15] which used an imposed $\overline{\mathcal{E}}$. Here $\chi \propto (1 + h_2)$. (b) Same as (a) but with $\zeta \sim 1$.
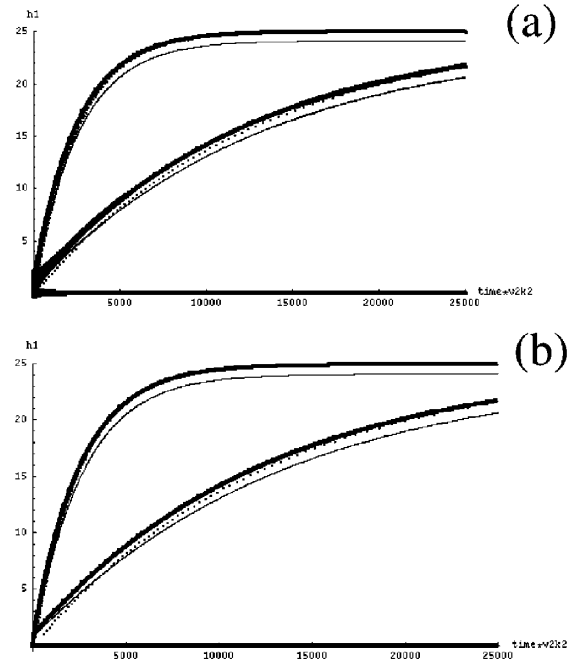
FIG. 2. (a) Same as Fig. 1(a) but for broader time range. (b) Same as (a) but for $\zeta = 1$. For this time range, the doublet and triplet solutions are indistinguishable. The dotted curves are fits to simulations [12]. The lines slightly below each of the thick lines are for $\chi \propto 1/(1 + k_1h_1/k_2)$ demonstrating the weak dependence on $\tilde{\beta}$.

$-1 < h_2 < 0$, $Q$ grows more negative and $h_1$ and $h_2$ grow with mutually opposite signs. As $h_2$ passes through $-1$ from above, $\partial_\tau Q$ changes sign immediately but $h_1$ continues to grow positive, albeit more slowly, until $Q$ changes sign. Then, $\partial_\tau h_1$ changes sign and $h_1$ decreases. But $\partial_\tau h_2$ changes sign when $\partial_\tau h_1$ does, so when $h_2$ eventually passes back through $-1$ from below, $\partial_\tau Q$ reverses sign again, and eventually $Q$ becomes negative and $h_1$ again grows. Large $R_M$ terms only weakly damp the oscillations. This describes what happens in Fig. 1(a). If instead, $\zeta \sim 1$, once $\partial_\tau Q$ is depleted by the growth of $h_2$, the $\zeta$ term of (16) takes over and $Q$ decays without oscillating. Then $h_1$ grows without oscillations [Fig. 1(b)].

*Discussion.*—Textbook kinematic MFD theories solve only the MFD equation itself [1]. Recent nonlinear approaches incorporating magnetic helicity evolution dynamically [15,16] solve a doublet: the MFD equation (or the $\partial_t H_1^M$ equation) and the total magnetic helicity evolution equation (the $\partial_t H_1^M + \partial_t H_2^M$ equation). The present paper solves a triplet: the MFD equation, the total magnetic helicity evolution equation, *and* the $\partial_t \overline{\mathcal{E}}$ equation. Only the present approach shows how the difference between kinetic and current helicities (the residual helicity) emerges as the MFD driver in a time-dependent theory. The residual helicity in turn couples to the total magnetic helicity evolution equation. The physical interpretation of the solutions for a closed system is that, as the large-scale helical field grows from MFD action, the small-scale magnetic helicity grows of the opposite sign. At early times, kinematic growth is unimpeded, and the large-scale field energy grows to $\overline{B}^2 \gtrsim (k_1/k_2)v_2^2$. Eventually, the small-scale magnetic helicity backreacts on the kinetic helicity, suppressing the growth rate to an $R_M$ dependent value. Ultimately $\overline{B}^2 \simeq (k_2/k_1)v_2^2$ at saturation. This picture also arises in the imposed $\overline{\mathcal{E}}$ doublet approach [15]. The approaches agree in the asymptotic $R_M$ dependent growth phase, matching simulations [12]. Oscillations can occur at early times only in the triplet approach, and only when triple correlations are assumed to be negligible (as in the first order smoothing approximation). The oscillations thus serve as a diagnostic for the MHD closure scheme and can be tested with future numerical experiments. For $\zeta = 1$, the oscillations disappear and the triplet and doublet approaches agree exactly. This corresponds to a closure in which triple correlations act as a damping term with damping time $\sim \tau_c$.

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