Harmonic Measure and Winding of Conformally Invariant Curves

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The exact joint multifractal distribution for the scaling and winding of the electrostatic potential lines near any conformally invariant scaling curve is derived in two dimensions. Its spectrum $f(\alpha, \lambda)$ gives the Hausdorff dimension of the points where the potential scales with distance r as $H \sim r^{\alpha}$ while the curve logarithmically spirals with a rotation angle $\varphi = \lambda \ln r$. It obeys the scaling law $f(\alpha, \lambda) = (1 + \lambda^2) f(\bar{\alpha}) - b\lambda^2$ with $\bar{\alpha} = \alpha/(1 + \lambda^2)$ and $b = (25 - c)/12$, and where $f(\alpha) \equiv f(\alpha, 0)$ is the pure harmonic measure spectrum, and c the conformal central charge. The results apply to $O(N)$ and Potts models, as well as to stochastic Löwner evolution.

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The geometric description of the random fractals arising in Nature is a fascinating subject. Outstanding among these fractals is the class of random clusters or curves arising in equilibrium critical phenomena, which are associated with fundamental ideas of *scale invariance*. In particular, in *two dimensions* (2D), statistical systems at their critical point are expected to produce *conformally invariant* (CI) fractal structures [1], with a Gibbs equilibrium weight invariant under all planar conformal maps. This leads to a *universal random geometry*. Prominent examples are the continuum scaling limits of random walks (RW), i.e., Brownian motion, self-avoiding walks (SAW), and critical percolation, Ising, or Potts clusters. A wealth of exact methods has been devised for their study: Coulomb gas, conformal field theory (CFT) [2], and quantum gravity methods [3,4]. Recently, rigorous probabilistic methods have also been developed, with the introduction of the stochastic Löwner evolution (SLE) process, which directly mimics the wandering of critical cluster boundaries in the scaling limit [5].

A refined way of accessing this random geometry is provided by a classical potential theory of electrostatic or diffusion fields near these random fractal boundaries, whose self-similarity is reflected in a *multifractal* (MF) spectrum describing the singularities of the potential, also called harmonic measure. In 2D, the first exact examples appeared for the universality class of random or self-avoiding walks and percolation clusters, which all possess the same harmonic MF spectrum [6] (see also [7]), in contradistinction to higher dimensions [8]. The general solution for the potential distribution near any CI fractal in 2D, obtained in [9], depends only on the so-called *central charge c*, the parameter labeling the universality class of the underlying CFT (see also [10,11]). This solution can be generalized to higher multifractal correlations, such as the joint distribution of potential on both sides of a simple scaling path [12].

The important question remains of the *geometry of the equipotential lines* near a random (CI) fractal curve. They are expected to wildly rotate, or wind, in a spiralling motion, that closely follows the boundary itself. The key geometrical object is here the *logarithmic spiral*, which is conformally invariant. The MF description should generalize to a *mixed* multifractal spectrum, accounting for *both scaling and winding* of the equipotentials [13].

In this Letter, we obtain the exact solution to this mixed MF spectrum for any random CI curve. In particular, it is shown to be related by a scaling law to the usual harmonic MF spectrum. We use conformal tools (fusing quantum gravity and Coulomb gas methods), which allow the description of Brownian paths interacting and winding with CI curves, thereby providing a probabilistic description of the potential map.

Harmonic measure and rotations.—Consider a single (CI) critical random cluster, generically called C. Let $\mathcal{H}(z)$ be the potential at the exterior point $z \in \mathbb{C}$, with Dirichlet boundary conditions $\mathcal{H}(w \in \partial C) = 0$ on the outer (simply connected) boundary ∂C of C, and $\mathcal{H}(w) = 1$ on a circle "at ∞," i.e., of a large radius scaling similar to the average size R of C. As is well known, $\mathcal{H}(z)$ is identical to the probability that a Brownian path started at z escapes to "∞" without having hit C.

Let us now consider the *degree with which the curves wind in the complex plane about point w* and call $\varphi(z) = \arg(z - w)$. In the scaling limit, the multifractal formalism [14–16], here generalized to take into account rotations [13], characterizes subsets $\partial C_{\alpha,\lambda}$ of boundary sites by a Hölder exponent α, and a rotation rate λ, such that their potential lines, respectively, scale and *logarithmically spiral* as

$$\mathcal{H}(z \to w \in \partial C_{\alpha,\lambda}) \approx r^{\alpha}, \qquad \varphi(z \to w \in \partial C_{\alpha,\lambda}) \approx \lambda \ln r, \tag{1}$$

in the limit $r = |z - w| \to 0$. The Hausdorff dimension

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$\dim(\partial C_{\alpha,\lambda}) = f(\alpha, \lambda)$ defines the mixed MF spectrum, which is CI since *under a conformal map both* α *and* λ *are locally invariant.*

One can also consider the *harmonic measure* $H(w, r)$, which is the integral of the Laplacian of $\mathcal{H}$ in a disk $B(w, r)$ of radius r centered at $w \in \partial C$, i.e., the boundary charge in that disk. It scales as r^α with the same exponent as in (1), and is also the probability that a Brownian path started at large distance R first hits the boundary in $B(w, r)$. Let $\varphi(w, r)$ be the associated winding angle of the path down to distance r from w. The *mixed* moments of H and e^φ, averaged over all realizations of C, are defined as

$$\mathcal{Z}_{n,p} = \left\langle \sum_{w\in\partial C_r} H^n(w, r) \exp[p\,\varphi(w, r)] \right\rangle \approx (r/R)^{\tau(n,p)}, \tag{2}$$

where the sum runs over the centers of a covering of the boundary by disks of radius r, and where n and p are real numbers. The scaling limit involves multifractal scaling exponents $\tau(n, p)$ which vary in a nonlinear way with n and p [13–16]. They obey the symmetric double Legendre transform

$$\begin{aligned} \alpha = \frac{\partial\tau}{\partial n}(n, p), \qquad \lambda = \frac{\partial\tau}{\partial p}(n, p), \\ f(\alpha, \lambda) = \alpha n + \lambda p - \tau(n, p), \\ n = \frac{\partial f}{\partial\alpha}(\alpha, \lambda), \qquad p = \frac{\partial f}{\partial\lambda}(\alpha, \lambda). \end{aligned} \tag{3}$$

Because of the ensemble average (2), values of $f(\alpha, \lambda)$ can become negative for some domains of α, λ.

Exact mixed multifractal spectra.—Each 2D conformally invariant random statistical system can be labeled by its *central charge c*, $c \le 1$ [1]. Our main result is the following exact scaling law:

$$\begin{aligned} f(\alpha, \lambda) = (1 + \lambda^2) f\left(\frac{\alpha}{1+\lambda^2}\right) - b\lambda^2, \\ b \equiv \frac{25 - c}{12} \ge 2, \end{aligned} \tag{4}$$

where $f(\alpha) \equiv f(\alpha, \lambda = 0)$ is the usual harmonic MF spectrum in the absence of prescribed winding, first obtained in [9], which can be recast as

$$f(\alpha) = \alpha + b - \frac{b\alpha^2}{2\alpha - 1}. \tag{5}$$

We thus arrive at the very simple formula,

$$f(\alpha, \lambda) = \alpha + b - \frac{b\alpha^2}{2\alpha - 1 - \lambda^2}. \tag{6}$$

Notice that by conformal symmetry $\sup_\lambda f(\alpha, \lambda) = f(\alpha, \lambda = 0)$; i.e., the most likely situation in the absence of prescribed rotation is the same as $\lambda = 0$, i.e., *winding free*. The domain of definition of the usual $f(\alpha)$ (5) is $\alpha \ge 1/2$ [9,17]; thus, for λ-spiralling points Eq. (4) gives

$$\alpha \ge \tfrac{1}{2}(1 + \lambda^2), \tag{7}$$

in agreement with a theorem by Beurling [13,17].

There is a geometrical meaning to the exponent α. For an angle with opening θ, $\alpha = \pi/\theta$; the quantity π/α can be regarded as a local generalized angle with respect to the harmonic measure. The geometrical MF spectrum of the boundary subset with such opening angle θ and spiralling rate λ reads from (6)

$$\hat{f}(\theta, \lambda) \equiv f\left(\alpha = \frac{\pi}{\theta}, \lambda\right) = \frac{\pi}{\theta} + b - b\frac{\pi}{2}\left(\frac{1}{\theta} + \frac{1}{\frac{2\pi}{1+\lambda^2} - \theta}\right).$$

As in (7), the domain of definition in the θ variable is $0 \le \theta \le \theta(\lambda)$, with $\theta(\lambda) = 2\pi/(1 + \lambda^2)$. The maximum is reached when the two frontier strands about point w locally collapse into a single λ spiral, whose inner opening angle is $\theta(\lambda)$ [17].

In the absence of prescribed winding ($\lambda = 0$), the maximum $D_{\mathrm{EP}} \equiv D_{\mathrm{EP}}(0) = \sup_\alpha f(\alpha, \lambda = 0)$ gives the dimension of the *external perimeter* (EP) of the fractal cluster, which is a *simple* curve without double points, and may differ from the full hull [9,18]. Its dimension reads [9] $D_{\mathrm{EP}} = \frac{1}{2}(1 + b) - \frac{1}{2}\sqrt{b(b-2)}$. This corresponds to typical values $\hat{\alpha} = \alpha(n = 0, p = 0)$ and $\hat{\theta} = \pi/\hat{\alpha} = \pi(3 - 2D_{\mathrm{EP}})$.

For spirals, the maximum value $D_{\mathrm{EP}}(\lambda) = \sup_\alpha f(\alpha, \lambda)$ still corresponds in the Legendre transform (3) to $n = 0$, and gives the dimension of the *subset of the external perimeter made of logarithmic spirals of type* λ. Owing to (4), we immediately get

$$D_{\mathrm{EP}}(\lambda) = (1 + \lambda^2) D_{\mathrm{EP}} - b\lambda^2. \tag{8}$$

This corresponds to scaled typical values $\hat{\alpha}(\lambda) = (1 + \lambda^2)\hat{\alpha}$, and $\hat{\theta}(\lambda) = \hat{\theta}/(1 + \lambda^2)$. Since $b \ge 2$ and $D_{\mathrm{EP}} \le 3/2$, the EP dimension decreases with spiralling rate, in a simple parabolic way.

Figure 1 displays typical multifractal functions $f(\alpha, \lambda; c)$. The example chosen, $c = 0$, corresponds to the cases of a SAW, or of a percolation EP, the scaling limits of which both coincide with the Brownian frontier [6,7]. The original singularity at $\alpha = \frac{1}{2}$ in the rotation-free MF functions $f(\alpha, 0)$, which describes boundary

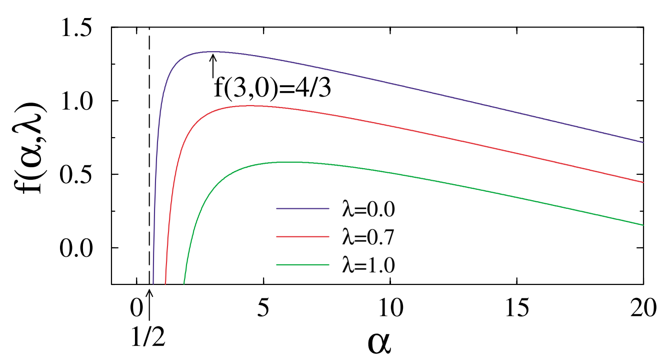

FIG. 1 (color). Universal multifractal spectrum $f(\alpha, \lambda)$ for $c = 0$ (Brownian frontier, percolation EP, and SAW), and for three different values of the spiralling rate λ. The maximum $f(3, 0) = 4/3$ is the Hausdorff dimension of the frontier.

points with a needle local geometry, is shifted for $\lambda \neq 0$ towards the minimal value (7). The right branch of $f(\alpha, \lambda)$ has a linear asymptote $\lim_{\alpha \to +\infty} f(\alpha, \lambda)/\alpha = -(1-c)/24$. Thus, the λ curves all become parallel for $\alpha \to +\infty$; i.e., $\theta \to 0^+$, corresponding to deep fjords where winding is easiest.

Limit multifractal spectra are obtained for $c = 1$, which exhibit *exact* examples of *left-sided* MF spectra, with a horizontal asymptote $f(\alpha \to +\infty, \lambda; c = 1) = \frac{3}{2} - \frac{1}{2}\lambda^2$ (Fig. 2). This corresponds to the frontier of a $Q = 4$ Potts cluster (i.e., the $\mathrm{SLE}_{\kappa=4}$), a universal random scaling curve, with the maximum value $D_{\mathrm{EP}} = 3/2$, and a vanishing typical opening angle $\hat{\theta} = 0$, i.e., the "ultimate Norway" where the EP is dominated by "*fjords*" everywhere [9,12].

Figure 3 displays the dimension $D_{\mathrm{EP}}(\lambda)$ as a function of the rotation rate λ, for various values of $c \leq 1$, corresponding to different statistical systems. Again, the $c = 1$ case shows the least decay with λ, as expected from the predominance of fjords there.

Conformal invariance and quantum gravity.—We now give the main lines of the derivation of exponents $\tau(n, p)$, hence $f(\alpha, \lambda)$, by generalized *conformal invariance*. By definition of the H measure, n *independent* Brownian paths $\mathcal{B}$, starting a small distance r away from a point w of the frontier ∂C, and diffusing without hitting ∂C, give a geometric representation of the nth moment, H^n, in Eq. (2) for n *integer*. Convexity yields analytic continuation for arbitrary n. Let us introduce an abstract (conformal) field operator $\Phi_{\partial C \wedge n}$ characterizing the presence of a vertex where n such Brownian paths and the cluster's frontier diffuse away from each other in a *mutually avoiding* configuration noted $\partial C \wedge n$ [6]; to this operator is associated a scaling dimension $x(n)$. To measure rotations as in moments (2), we have to consider expectation values with insertion of the mixed operator,

$$\Phi_{\partial C \wedge n} e^{p \arg(\partial C \wedge n)} \longrightarrow x(n, p) = \tau(n, p) + 2, \qquad (9)$$

where $\arg(\partial C \wedge n)$ is the winding angle common to the frontier and to the Brownian paths, and where $x(n, p)$ is the *scaling dimension*, directly related to $\tau(n, p)$ [6].

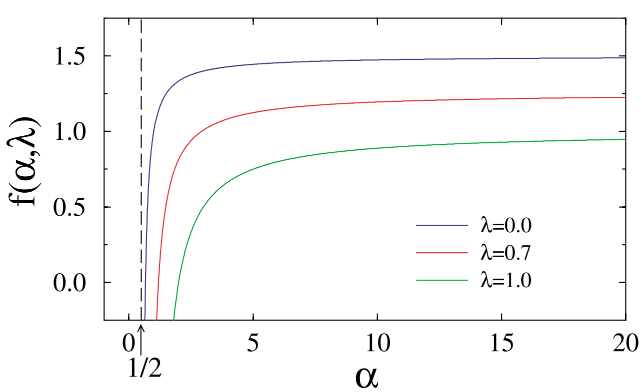

FIG. 2 (color). Left-sided multifractal spectra $f(\alpha, \lambda)$ for the limit case $c = 1$ (frontier of a $Q = 4$ Potts cluster or $\mathrm{SLE}_{\kappa=4}$).

One has $x(n, p = 0) = x(n)$, and $\tau(n, p = 0) \equiv \tau(n) = x(n) - 2$.

Let us now use a fundamental mapping between the CFT in the *plane* $\mathbb{R}^2$ and the CFT on a fluctuating abstract random Riemann surface, i.e., in the presence of *2D quantum gravity* (QG) [3]. Two universal functions U and V, acting on scaling dimensions, describe this map:

$$U(x) = x\frac{x - \gamma}{1 - \gamma}, \qquad V(x) = \frac{1}{4}\frac{x^2 - \gamma^2}{1 - \gamma}, \qquad (10)$$

with $V(x) \equiv U[\frac{1}{2}(x + \gamma)]$ [6,9]. The parameter γ is the solution of $c = 1 - 6\gamma^2(1 - \gamma)^{-1}$, $\gamma \leq 0$.

For the purely harmonic exponents $x(n)$, describing the mutually avoiding set $\partial C \wedge n$, we have [6,9]

$$x(n) = 2V[2U^{-1}(\tilde{x}_1) + U^{-1}(n)], \qquad (11)$$

where $U^{-1}(x)$ is the positive inverse of U,

$$2U^{-1}(x) = \sqrt{4(1 - \gamma)x + \gamma^2} + \gamma.$$

In (11), the arguments $\tilde{x}_1$ and n are, respectively, the *boundary* scaling dimensions (b.s.d.) of the simple path S_1 representing a semi-infinite random frontier (such that $\partial C \equiv S_1 \wedge S_1$), and of the packet of n Brownian paths, both diffusing into the upper *half-plane* $\mathbb{H}$. The function U^{-1} maps these half-plane b.s.d. to the corresponding b.s.d. in quantum gravity, the *linear combination* of which gives, still in QG, the b.s.d. of the *mutually avoiding set* $\partial C \wedge n = (\wedge S_1)^2 \wedge n$. The function V finally maps the latter b.s.d. into the scaling dimension in $\mathbb{R}^2$. The path b.s.d. $\tilde{x}_1$ obeys $U^{-1}(\tilde{x}_1) = (1 - \gamma)/2$ [9].

It is now useful to consider k semi-infinite random paths S_1, joined at a single vertex in a *mutually avoiding star* configuration $S_k = \overbrace{S_1 \wedge S_1 \wedge \cdots S_1}^{k} = (\wedge S_1)^k$. Its scaling dimension can be obtained from the same b.s.d. additivity rule in quantum gravity, as in (11) [6,9]

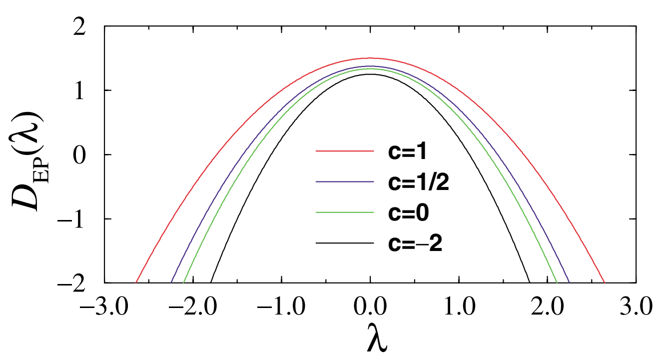

FIG. 3 (color). Dimensions $D_{\mathrm{EP}}(\lambda)$ of the external frontiers as a function of rotation rate. The curves are indexed by the central charge c, and correspond, respectively, to the following: loop-erased RW ($c = -2$; SLE_2); Brownian or percolation external frontiers, and self-avoiding walk ($c = 0$; $\mathrm{SLE}_{8/3}$); Ising clusters ($c = \frac{1}{2}$; SLE_3); $Q = 4$ Potts clusters ($c = 1$; SLE_4).

$$x(S_k) = 2V[k\, U^{-1}(\tilde{x}_1)]. \qquad (12)$$

The scaling dimensions (11) and (12) coincide when

$$x(n) = x(S_{k(n)}), \qquad k(n) = 2 + \frac{U^{-1}(n)}{U^{-1}(\tilde{x}_1)}. \qquad (13)$$

Thus, we state the *scaling star equivalence* $\partial C \wedge n \Longleftrightarrow S_{k(n)}$, *of two simple paths* S_1 *avoiding n Brownian motions to* $k(n)$ *simple paths in a mutually avoiding star configuration*, an equivalence which will also play an essential role in the complete rotation spectrum (9).

Rotation scaling exponents.—The Gaussian distribution of the winding angle about the *extremity* of a scaling path, such as S_1, was derived in [19], using exact Coulomb gas methods. The argument can be generalized to the winding angle of a star S_k about its center [20], where one finds that the angular variance is reduced by a factor $1/k^2$ (see also [21]). The scaling dimension associated with the rotation scaling operator $\Phi_{S_k} e^{p \arg(S_k)}$ is found by analytic continuation of the Fourier transforms evaluated there [20]:

$$x(S_k; p) = x(S_k) - \frac{2}{1-\gamma}\frac{p^2}{k^2},$$

i.e., is given by a quadratic shift in the star scaling exponent. To calculate the scaling dimension (9), it suffices to use the star equivalence (13) above to show that

$$x(n, p) = x(S_{k(n)}; p) = x(n) - \frac{2}{1-\gamma}\frac{p^2}{k^2(n)},$$

which is the key to our problem. Using Eqs. (13), (11), and (10) gives the useful identity:

$$\tfrac{1}{8}(1-\gamma)k^2(n) = x(n) - 2 + b,$$

with $b = \frac{1}{2}[(2-\gamma)^2]/(1-\gamma) = \frac{25-c}{12}$. Recalling (9), we arrive at the multifractal result:

$$\tau(n, p) = \tau(n) - \frac{1}{4}\frac{p^2}{\tau(n) + b}, \qquad (14)$$

where $\tau(n) = x(n) - 2$ corresponds to the purely harmonic spectrum with no prescribed rotation.

Legendre transform.—The structure of the full τ function (14) leads by a formal Legendre transform (3) directly to the identity

$$f(\alpha, \lambda) = (1 + \lambda^2) f(\bar{\alpha}) - b\lambda^2,$$

where $f(\bar{\alpha}) \equiv \bar{\alpha} n - \tau(n)$, with $\bar{\alpha} = d\tau(n)/dn$, is the purely harmonic MF function. It depends on the natural reduced variable $\bar{\alpha}$ *à la* Beurling $\{\bar{\alpha} \in [\frac{1}{2}, +\infty)\}$

$$\bar{\alpha} \equiv \frac{\alpha}{1+\lambda^2} = \frac{dx}{dn}(n) = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{b}{2n + b - 2}},$$

whose expression is found explicitly from (11). Whence Eq. (4), **QED**.

O(N) and Potts models, SLE$_\kappa$.—Our results apply to the critical $O(N)$ loop model, or to the EP of critical Fortuin-Kasteleyn (FK) clusters in the Q-Potts model, all described in terms of Coulomb gas with some coupling constant g [2]. SLE$_\kappa$ paths also describe cluster frontiers or hulls. One has the correspondence $\kappa = 4/g$, with a central charge $c = (3-2g)(3-2g') = \frac{1}{4}(6-\kappa)(6-\kappa')$, symmetric under the *duality* $gg' = 1$ or $\kappa\kappa' = 16$. This duality gives FK-EP as some simple random $O(N)$ loops, or, equivalently, the SLE$_{\kappa' \leq 4}$ as the simple frontier of the SLE$_{\kappa \geq 4}$ [9,12].

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