Configuration of Separability and Tests for Multipartite Entanglement in Bell-Type Experiments

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We derive tight quadratic inequalities for all kinds of hybrid separable-inseparable n-particle density operators on an arbitrary dimensional space. This methodology enables us to derive a tight quadratic inequality as tests for full n-partite entanglement in various Bell-type correlation experiments on the systems that may not be identified as a collection of qubits, e.g., those involving photons measured by incomplete detectors. It is also proved that when the two measured observables are assumed to precisely anticommute, a stronger quadratic inequality can be used as a witness of full n-partite entanglement.

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Since the 1980s, it has been a problem how to confirm multipartite entanglement experimentally. Recently, we have been given precious experimental data by efforts of experimentalists [1,2]. Proper analysis of these experimental data then becomes necessary, and as a result of such analysis [3], the experimental data obtained by Pan and co-workers [2] confirms the existence of genuinely three-particle entanglement under the assumption that proper observables are measured in the experiment. However, it was discussed [4] that for other experimental data there is a loophole problem in confirming three-particle entanglement, and the loophole problem remains unresolved. This means that there have not been enough discussions about what kind of data are needed for confirming multipartite entanglement.

There have been many researches on the problem, providing inequalities for functions of experimental correlations [3–10]. Among them, assuming k-partite split of the system [11] without assuming a specific partition, Werner and Wolf derived an upper bound $2^{(n-k)/2}$ for expectation values of n-particle Bell-Mermin operators $\mathcal{B}, \mathcal{B}'$ [10,12] under the assumption that suitable partial transposes of the density operator are positive [10]. The inequality derived by Werner *et al.* is useful because it tells us a number k such that the given state is at most k separable [11,13].

Recently, Uffink introduced a nonlinear inequality aimed at giving stronger tests for full n-partite entanglement than previous formulas. For qubit systems, Uffink has derived [9] a tight quadratic inequality for the states where one qubit is not entangled with any other qubit; namely, the states written as a convex sum over the states of form $\rho_1 \otimes \rho_{2,\ldots,n}$.

In most of the real experiments, we have to deal with higher dimensional systems rather than qubit systems. For example, when polarizations of photons from a nonideal source are measured by imperfect detectors, it is difficult to claim strictly that the observed correlations are obtained by measuring subsystems with only two-dimensional Hilbert spaces, due to the ambiguity in the number of photons. The arguments about higher dimensional systems will thus be necessary in order to establish tests applicable to real experiments without making auxiliary assumptions as to the dimension of the measured space or as to measured observables.

In deriving a witness of full n-partite entanglement, it should be ensured that the witness rules out all hybrid separable-inseparable states except genuine fully n-partite entangled states. The hybrid separable-inseparable states are depicted as follows: Consider a partition of n-particle system $\{1, 2, \ldots, n\}$ into k nonempty and disjoint subsets $\alpha_1, \ldots, \alpha_k$, where $\sum_{i=1}^{k} |\alpha_i| = n$, to which we refer as a k-partite split of the system [11] . Let us now consider the density operators ρ on $\mathcal{H} = \bigotimes_{j=1}^{n} \mathcal{H}_j$, where $\mathcal{H}_j$ represents the Hilbert space with respect to particle j. Then all hybrid separable-inseparable states with respect to partition $\alpha_1, \ldots, \alpha_k$ can be written as

$$\rho = \sum_l p_l(\otimes_{i=1}^{k} \rho_l^{\alpha_i}), \qquad \left(p_l \geq 0, \sum_l p_l = 1\right), \qquad (1)$$

where $\rho_l^{\alpha_i}, \forall l$ are the density operators on the partial Hilbert space $\bigotimes_{j\in\alpha_i} \mathcal{H}_j$. States (1) are called k separable with respect to a partition $\alpha_1, \ldots, \alpha_k$.

In this paper, we derive the optimal upper bound of $\langle\mathcal{B}\rangle^2 + \langle\mathcal{B}'\rangle^2$ for any partition of the systems $\alpha_1, \ldots, \alpha_k$ of an arbitrary dimensional space. It turns out that the optimal upper bound depends only on two parameters k and m, and not on the detailed configuration of the partition, where m is the number of particles which are not entangled with any other particles. The maximum is given by $2^{n+m-2k+1}$. Using this maximum, we genuinely prove that the optimal upper bound that is utilizable to confirm full n-partite entanglement ($n \geq 3$) of an arbitrary dimensional system is 2^{n-2}. Later, we show that if an auxiliary assumption as to measured observables is allowed, a stronger quadratic inequality can be used as a witness of full n-partite entanglement.

In what follows, we derive tight quadratic inequalities for hybrid separable-inseparable states with respect to partition $\alpha_1, \ldots, \alpha_k$ of an arbitrary dimensional space. It is assumed that a measurement with two outcomes, ± 1, is performed on each particle. Such a measurement is generally described by a positive-operator-valued measure (POVM), $\{F_+, F_-\}$, $F_+ + F_- = \mathbf{1}$, $F_+, F_- \geq 0$, and the corresponding observable is given by a Hermitian operator $A = F_+ - F_-$, which has a spectrum in $[-1, 1]$. We assume that for each particle j, either of two such observables A_j or A'_j is chosen, where $-\mathbf{1} \leq A_j, A'_j \leq \mathbf{1}$, $\forall j$.

The Bell-Mermin operators take a simple form when we view on a complex plane using a function $f(x, y) = \frac{1}{\sqrt{2}} e^{-i\pi/4}(x + iy)$, $x, y \in \mathbf{R}$. Note that this function is invertible, as $x = \mathrm{Re}f - \mathrm{Im}f$, $y = \mathrm{Re}f + \mathrm{Im}f$. The Bell-Mermin operators $\mathcal{B}_{\mathbf{N}_n}$ and $\mathcal{B}'_{\mathbf{N}_n}$ are defined by [10,12]

$$f(\mathcal{B}_{\mathbf{N}_n}, \mathcal{B}'_{\mathbf{N}_n}) = \otimes_{j=1}^{n} f(A_j, A'_j), \tag{2}$$

where $\mathbf{N}_n = \{1, 2, \ldots, n\}$. We also define $\mathcal{B}_\alpha$ for any subset $\alpha \subset \mathbf{N}_n$ by

$$f(\mathcal{B}_\alpha, \mathcal{B}'_\alpha) = \otimes_{j\in\alpha} f(A_j, A'_j). \tag{3}$$

It is easy to see, when $\alpha, \beta (\subset \mathbf{N}_n)$ are disjoint, that

$$f(\mathcal{B}_{\alpha\cup\beta}, \mathcal{B}'_{\alpha\cup\beta}) = f(\mathcal{B}_\alpha, \mathcal{B}'_\alpha) \otimes f(\mathcal{B}_\beta, \mathcal{B}'_\beta), \tag{4}$$

which leads to

$$\begin{aligned} \mathcal{B}_{\alpha\cup\beta} &= 1/2(\mathcal{B}_\alpha\mathcal{B}'_\beta + \mathcal{B}'_\alpha\mathcal{B}_\beta) + 1/2(\mathcal{B}_\alpha\mathcal{B}_\beta - \mathcal{B}'_\alpha\mathcal{B}'_\beta), \\ \mathcal{B}'_{\alpha\cup\beta} &= 1/2(\mathcal{B}_\alpha\mathcal{B}'_\beta + \mathcal{B}'_\alpha\mathcal{B}_\beta) - 1/2(\mathcal{B}_\alpha\mathcal{B}_\beta - \mathcal{B}'_\alpha\mathcal{B}'_\beta). \end{aligned} \tag{5}$$

First, we prove that the following inequality proposed by Uffink for qubit systems is also valid for an arbitrary dimensional system:

$$\langle\mathcal{B}_\alpha\rangle^2 + \langle\mathcal{B}'_\alpha\rangle^2 \leq 2^{|\alpha|-1}, \qquad (|\alpha| \geq 2). \tag{6}$$

In order to see this, we use the following lemma.

Lemma: Let $-\mathbf{1} \leq X_i, X'_i \leq \mathbf{1}$ be Hermitian operators ($i = 1, 2$), and define Y, Y' as follows:

$$f(Y, Y') = f(X_1, X'_1) \otimes f(X_2, X'_2). \tag{7}$$

Then

$$\langle Y\rangle^2 + \langle Y'\rangle^2 \leq 2. \tag{8}$$

The lemma is proven in the following way: Note that

$$\begin{aligned} Y &= (1/2)\{X_1(X_2 + X'_2) + X'_1(X_2 - X'_2)\}, \\ Y' &= (1/2)\{X'_1(X'_2 + X_2) + X_1(X'_2 - X_2)\}, \end{aligned} \tag{9}$$

and let B_θ be $Y\cos\theta + Y'\sin\theta$. Let us derive the maximum value of $\mathrm{tr}[\rho B_\theta]$. Note that $\mathrm{tr}[\rho B_\theta]$ is a linear function of each X_i or X'_i, keeping ρ fixed. Therefore we may consider only the set of extremal points in the convex set of Hermitian operators with $-\mathbf{1} \leq X \leq \mathbf{1}$. Hence we may assume $X_i^2 = X_i'^2 = \mathbf{1} (i = 1, 2)$. We thus have

$$\begin{aligned} Y^2 = Y'^2 &= \mathbf{1} - (1/4)[X_1, X'_1] \otimes [X_2, X'_2] \\ &= \mathbf{1} + A_1^- \otimes A_2^-, \end{aligned} \tag{10}$$

where $A_i^- = (i/2)[X_i, X'_i]$ (A_i^- are Hermitian operators) and

$$\{Y, Y'\} = (1/2)\{X_1, X'_1\} \otimes \{X_2, X'_2\} = 2A_1^+ \otimes A_2^+, \tag{11}$$

where $A_i^+ = (1/2)\{X_i, X'_i\}$ (A_i^+ are Hermitian operators). Then we obtain

$$B_\theta^2 = \mathbf{1} + A_1^- \otimes A_2^- + \sin 2\theta A_1^+ \otimes A_2^+. \tag{12}$$

This implies

$$\begin{aligned} \mathrm{tr}[\rho B_\theta^2] \leq \max\{1 &+ \|A_1^+ \otimes A_2^+ + A_1^- \otimes A_2^-\|, 1 \\ &+ \|A_1^+ \otimes A_2^+ - A_1^- \otimes A_2^-\|\}, \end{aligned} \tag{13}$$

where $\|\cdot\|$ means the operator norm. Note that

$$\begin{aligned} A_1^+ \otimes A_2^+ \pm A_1^- \otimes A_2^- = (1/2)\{&(A_1^+ + iA_1^-) \otimes (A_2^+ \mp iA_2^-) \\ &+ (A_1^+ - iA_1^-) \\ &\otimes (A_2^+ \pm iA_2^-)\}, \end{aligned} \tag{14}$$

and

$$\begin{aligned} A_1^+ \otimes A_2^+ + A_1^- \otimes A_2^- = (1/2)(&X'_1X_1 \otimes X_2X'_2 + X_1X'_1 \\ &\otimes X'_2X_2). \end{aligned} \tag{15}$$

According to relationships such as $\|X'_1X_1\| = \|(X_1X'_1)(X'_1X_1)\|^{1/2} = \|\mathbf{1}\|^{1/2} = 1$, we get

$$\|A_1^+ \otimes A_2^+ + A_1^- \otimes A_2^-\| \leq 1. \tag{16}$$

Similarly, we also get

$$\begin{aligned} \|A_1^+ \otimes A_2^+ - A_1^- \otimes A_2^-\| = (1/2)\|&X'_1X_1 \otimes X'_2X_2 + X_1X'_1 \\ &\otimes X_2X'_2\| \leq 1. \end{aligned} \tag{17}$$

Therefore we have $|\mathrm{tr}[\rho B_\theta]|^2 \leq \mathrm{tr}[\rho B_\theta^2] \leq 2$ by the variance inequality. Now by taking

$$\cos\theta = \frac{\langle Y\rangle}{\sqrt{\langle Y\rangle^2 + \langle Y'\rangle^2}}, \qquad \sin\theta = \frac{\langle Y'\rangle}{\sqrt{\langle Y\rangle^2 + \langle Y'\rangle^2}}, \tag{18}$$

we obtain $\langle Y\rangle^2 + \langle Y'\rangle^2 \leq 2$, Q.E.D.

Let us consider a set $\alpha \subset \mathbf{N}_n$, where $|\alpha| \geq 2$. Let γ be $\alpha\backslash\{j\}$, where $j \in \alpha$. Then, from Eq. (4), we have

$$f(\mathcal{B}_\alpha, \mathcal{B}'_\alpha) = f(\mathcal{B}_\gamma, \mathcal{B}'_\gamma) \otimes f(A_j, A'_j). \tag{19}$$

It has been known that the maximum of $\langle\mathcal{B}_\gamma\rangle$ is $2^{(|\gamma|-1)/2}$ [10]. Noting that $f(cx, cy) = cf(x, y)$, $c \in \mathbf{R}$ and $-\mathbf{1} \leq 2^{-(|\gamma|-1)/2}\mathcal{B}_\gamma \leq \mathbf{1}$, according to the lemma by taking $X_1 = 2^{-(|\gamma|-1)/2}\mathcal{B}_\gamma$, $X'_1 = 2^{-(|\gamma|-1)/2}\mathcal{B}'_\gamma$, $X_2 = A_j$, and $X'_2 = A'_j$, we obtain the quadratic inequality

$$\langle\mathcal{B}_\alpha\rangle^2 + \langle\mathcal{B}'_\alpha\rangle^2 \leq 2^{|\alpha|-1}, \qquad (|\alpha| \geq 2), \tag{20}$$

where we used $|\alpha| = |\gamma| + 1$.

Next, we calculate an upper bound of $\langle\mathcal{B}_{\mathbf{N}_n}\rangle^2 + \langle\mathcal{B}'_{\mathbf{N}_n}\rangle^2$ for states of the form $\otimes_{i=1}^{k}\rho^{\alpha_i}$. From Eq. (5), we have

$$\begin{aligned} \langle\mathcal{B}_{\alpha\cup\beta}\rangle^2 + \langle\mathcal{B}'_{\alpha\cup\beta}\rangle^2 = (1/2)(&\langle\mathcal{B}_\alpha\mathcal{B}'_\beta + \mathcal{B}'_\alpha\mathcal{B}_\beta\rangle^2 \\ &+ \langle\mathcal{B}_\alpha\mathcal{B}_\beta - \mathcal{B}'_\alpha\mathcal{B}'_\beta\rangle^2). \end{aligned} \tag{21}$$

Using Eq. (21), we obtain

$$\begin{aligned}\langle\mathcal{B}_{\mathbf{N}_n}\rangle^2+\langle\mathcal{B}'_{\mathbf{N}_n}\rangle^2&=(1/2)(\langle\mathcal{B}_{\alpha_1}\mathcal{B}'_{\mathbf{N}_n\backslash\alpha_1}+\mathcal{B}'_{\alpha_1}\mathcal{B}_{\mathbf{N}_n\backslash\alpha_1}\rangle^2+\langle\mathcal{B}_{\alpha_1}\mathcal{B}_{\mathbf{N}_n\backslash\alpha_1}-\mathcal{B}'_{\alpha_1}\mathcal{B}'_{\mathbf{N}_n\backslash\alpha_1}\rangle^2)\\&=(1/2)\{(\langle\mathcal{B}_{\alpha_1}\rangle\langle\mathcal{B}'_{\mathbf{N}_n\backslash\alpha_1}\rangle+\langle\mathcal{B}'_{\alpha_1}\rangle\langle\mathcal{B}_{\mathbf{N}_n\backslash\alpha_1}\rangle)^2+(\langle\mathcal{B}_{\alpha_1}\rangle\langle\mathcal{B}_{\mathbf{N}_n\backslash\alpha_1}\rangle-\langle\mathcal{B}'_{\alpha_1}\rangle\langle\mathcal{B}'_{\mathbf{N}_n\backslash\alpha_1}\rangle)^2\}\\&=(1/2)(\langle\mathcal{B}_{\alpha_1}\rangle^2+\langle\mathcal{B}'_{\alpha_1}\rangle^2)(\langle\mathcal{B}_{\mathbf{N}_n\backslash\alpha_1}\rangle^2+\langle\mathcal{B}'_{\mathbf{N}_n\backslash\alpha_1}\rangle^2).\end{aligned}\qquad(22)$$

Repeating this, we obtain

$$\langle\mathcal{B}_{\mathbf{N}_n}\rangle^2+\langle\mathcal{B}'_{\mathbf{N}_n}\rangle^2=(1/2)^{k-1}\prod_{i=1}^{k}(\langle\mathcal{B}_{\alpha_i}\rangle^2+\langle\mathcal{B}'_{\alpha_i}\rangle^2).\qquad(23)$$

Without loss of generality, we assume that $|\alpha_i|=1$ for $1\le i\le m$ and $|\alpha_i|\ge 2$ for $m+1\le i\le k$. Applying $\langle\mathcal{B}_{\alpha_i}\rangle^2+\langle\mathcal{B}'_{\alpha_i}\rangle^2\le 2$ for $|\alpha_i|=1$ and Eq. (6), we obtain

$$\begin{aligned}\langle\mathcal{B}_{\mathbf{N}_n}\rangle^2+\langle\mathcal{B}'_{\mathbf{N}_n}\rangle^2&\le\prod_{i=m+1}^{k}2^{(|\alpha_i|-1)}(1/2)^{k-m-1}\\&=2^{n+m-2k+1},\end{aligned}\qquad(24)$$

where we used $\sum_{i=m+1}^{k}(|\alpha_i|-1)=(n-m)-(k-m)$. We then conclude [14] that, for any state ρ that is k separable with respect to $\alpha_1,\ldots,\alpha_k$,

$$(\mathrm{tr}[\rho\mathcal{B}_{\mathbf{N}_n}])^2+(\mathrm{tr}[\rho\mathcal{B}'_{\mathbf{N}_n}])^2\le 2^{n+m-2k+1}.\qquad(25)$$

The maximum depends only on two parameters k and m but not on the detailed configuration of the partition. Clearly, the bound (25) is optimal.

The inequality for testing full n-partite entanglement for $n\ge 3$ is obtained by maximizing the right-hand side of (25) under the condition $k\ge 2$. Noting that $m\le k-1$ when $k<n$, we obtain

$$\langle\mathcal{B}_{\mathbf{N}_n}\rangle^2+\langle\mathcal{B}'_{\mathbf{N}_n}\rangle^2\le 2^{n-2}.\qquad(26)$$

Violations of the inequality (26) imply full n-partite entanglement.

For multiqubit systems, Uffink considered the case of partitions of the form $\{1\},\{2,\ldots,n\}$, and has presented the quadratic inequality (26) for testing whether n-particle states are fully entangled [9]. In what we should pay attention to, we have to check that for all hybrid separable-inseparable states except genuine fully entangled states, the optimal upper bounds are smaller than or equal to 2^{n-2}, in order to ensure that the relation (26) can be used as tests for full n-partite entanglement. In this point, we genuinely proved that the violations of the relation (26) are sufficient for confirming fully n-partite entangled states. We have also proven that the relation (26) can be derived not only for multiqubit systems but also for higher dimensional systems.

The inequality (25) also implies

$$|\mathrm{tr}[\rho\mathcal{B}_{\mathbf{N}_n}]|\le 2^{(n+m-2k+1)/2}.\qquad(27)$$

It is known that $|\mathrm{tr}[\rho\mathcal{B}_{\mathbf{N}_n}]|\le 1$ when the system is fully separable [10]. Hence we obtain an upper bound [15]

$$|\mathrm{tr}[\rho\mathcal{B}_{\mathbf{N}_n}]|\le\begin{cases}2^{(n+m-2k+1)/2} & k<n\\ 1 & k=n.\end{cases}\qquad(28)$$

According to Eq. (5), the equality of the relation (28) holds when $\langle\mathcal{B}_{\alpha_i}\rangle=\langle\mathcal{B}'_{\alpha_i}\rangle=1$ for $1\le i\le m$, $\langle\mathcal{B}_{\alpha_i}\rangle=\langle\mathcal{B}'_{\alpha_i}\rangle=2^{(|\alpha_i|-2)/2}$ for $m+1\le i\le k-1$, and $\langle\mathcal{B}_{\alpha_k}\rangle=2^{(|\alpha_k|-1)/2}$. We can find a state and Hermitian operators $-\mathbf{1}\le A_j,A'_j\le\mathbf{1}$ that satisfy the above relations [16]. Hence the bound (28) is optimal.

For partitions of the form $\{1\},\{2\},\ldots,\{m\},\{m+1,\ldots,n\}(m\le n-1)$, the relation (28) leads to the result of Gisin and Bechmann-Pasquinucci [6], i.e., the bound $|\langle\mathcal{B}_{\mathbf{N}_n}\rangle|\le 2^{(n-m-1)/2}$. Noting that $m\le k-1$ when $k<n$, the relation (28) leads to the result of Werner and Wolf [10], i.e., $|\langle\mathcal{B}_{\mathbf{N}_n}\rangle|\le 2^{(n-k)/2}$ by taking the maximum over m with fixed k. Collins *et al.* considered the cases for partitions of the form $\{1\},\{2\},\{3,4\}$ or $\{1,2\},\{3,4\}$ or $\{1\},\{2,3,4\}$ and presented the bounds as $\sqrt{2},\sqrt{2},2$, respectively [7]. These bounds are also derived from the relation (28).

So far, we derived the threshold value (i.e., 2^{n-2}) of $\langle\mathcal{B}_{\mathbf{N}_n}\rangle^2+\langle\mathcal{B}'_{\mathbf{N}_n}\rangle^2$ for use as a full n-partite entanglement witness over all observables satisfying $-\mathbf{1}\le A_j,A'_j\le\mathbf{1}$. Now, let us consider an additional assumption that the two measured observables anticommute, i.e., $\{A_j,A'_j\}=\mathbf{0}\,\forall j$. This assumption is approximately fulfilled within the accuracy of the measurement apparatus in the common experimental situations, e.g., when we measure Pauli operators σ_x and σ_y for each particle. With this assumption, the threshold value of $\langle\mathcal{B}_{\mathbf{N}_n}\rangle^2+\langle\mathcal{B}'_{\mathbf{N}_n}\rangle^2$ becomes as small as 2^{n-3} as is shown below. This implies that we can use a stronger quadratic inequality as tests for full n-partite entanglement in this case.

Suppose that $\{A_j,A'_j\}=\mathbf{0}$ and $-\mathbf{1}\le A_j,A'_j\le\mathbf{1},\,\forall j$. Let us take $A_{j\theta}=A_j\cos\theta+A'_j\sin\theta$, and derive the maximum value of $\mathrm{tr}[\rho A_{j\theta}]$. Since we are interested only in the maximum, we may assume $A_j^2=A_j'^2=\mathbf{1}$. Then we get $A_{j\theta}^2=\mathbf{1}+(1/2)\{A_j,A'_j\}\sin 2\theta=\mathbf{1}$. The variance inequality leads to $|\mathrm{tr}[\rho A_{j\theta}]|^2\le\mathrm{tr}[\rho A_{j\theta}^2]=1$. Now take $\cos\theta=\langle A_j\rangle/\sqrt{\langle A_j\rangle^2+\langle A'_j\rangle^2}$, $\sin\theta=\langle A'_j\rangle/\sqrt{\langle A_j\rangle^2+\langle A'_j\rangle^2}$, then we get $\langle A_j\rangle^2+\langle A'_j\rangle^2\le 1\,\forall j$. This means that the relation (6) holds even for $|\alpha|=1$. Hence we obtain

$$\langle\mathcal{B}_{\alpha}\rangle^2+\langle\mathcal{B}'_{\alpha}\rangle^2\le 2^{|\alpha|-1},\qquad(|\alpha|\ge 1).\qquad(29)$$

Similar to the argument that derives (25), applying the relation (29), we conclude

$$(\mathrm{tr}[\rho \mathcal{B}_{\mathbf{N}_n}])^2 + (\mathrm{tr}[\rho \mathcal{B}'_{\mathbf{N}_n}])^2 \le 2^{n-2k+1}. \qquad (30)$$

The inequality for testing full n-partite entanglement is obtained by maximizing the right-hand side of (30) under the condition $k \ge 2$. We obtain [17]

$$\langle \mathcal{B}_{\mathbf{N}_n} \rangle^2 + \langle \mathcal{B}'_{\mathbf{N}_n} \rangle^2 \le 2^{n-3}. \qquad (31)$$

We give an example that the relation (31) is stronger than (26) as a witness of full n-partite entanglement for multiqubit systems. We assume that $A_j = \vec{a}_j \cdot \vec{\sigma}, A'_j = \vec{a}'_j \cdot \vec{\sigma}$, where $\vec{a}_j$ and $\vec{a}'_j$ are normalized vectors in $\mathbf{R}^3$ and $\vec{\sigma}$ is the vector of Pauli matrices, i.e., $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. The condition $\{A_j, A'_j\} = \mathbf{0}$ leads to $\vec{a}_j \cdot \vec{a}'_j = 0$. Let us consider the following multiqubit states [11]:

$$\rho = x|\Phi_n\rangle\langle\Phi_n| + \frac{1-x}{2^n} I, \qquad (32)$$

where I is the identity operator for the 2^n-dimensional space and $|\Phi_n\rangle$ is an n-qubit GHZ state [18], i.e.,

$$|\Phi_n\rangle = \frac{1}{\sqrt{2}}(|+_1, +_2, \ldots, +_n\rangle + |-_1, -_2, \ldots, -_n\rangle). \qquad (33)$$

It is easy to see that the maximum of $\langle \mathcal{B}_{\mathbf{N}_n} \rangle^2 + \langle \mathcal{B}'_{\mathbf{N}_n} \rangle^2$ is $2^{n-1}x^2$ with $\vec{a}_j \cdot \vec{a}'_j = 0 \forall j$ (see [19]). Hence, assuming that x is in the range of

$$\frac{1}{2} < x \le \frac{1}{\sqrt{2}}, \qquad (34)$$

we can confirm full n-partite entanglement from (31), which cannot be confirmed by (26). Hence if the measurement setups are precisely chosen as $\{A_j, A'_j\} = \mathbf{0} \forall j$, then one can use a stronger inequality as tests for full n-partite entanglement in comparison with the relation (26).

In real experimental situations, we cannot claim that $\{A_j, A'_j\} = \mathbf{0}$ with arbitrary precision. The relevance of the bounds claiming full n-partite entanglement assuming that $|\langle\{A_j, A'_j\}\rangle| \le \epsilon$, where ϵ means experimental errors, would be worth further investigations.

In summary, we have derived the quadratic inequality that is utilizable to test full n-partite entanglement not only for qubit systems but also for higher dimensional systems. This helps the analysis of experimental data in realistic situations. We have also shown that when the two measured observables are assumed to precisely anticommute, we can use a stronger quadratic inequality as a witness of full n-partite entanglement in correlation experiments.

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[13] We can estimate the minimum number of entangled particles as $n/k, (n/k \in \mathbf{N})$ or $[n/k] + 1, (n/k \notin \mathbf{N})$, when we are given the maximum positive integer k which is allowed by experimental data.

[14] As have been discussed in Ref. [9], as to any Hermitian operator $\mathcal{O}$, the expression $(\mathrm{tr}[\rho\mathcal{O}])^2$ is a convex function of ρ, and it is sufficient to consider only the states $\otimes_{i=1}^k \rho^{\alpha_i}$ to obtain the optimal upper bound for states (1).

[15] One can rewrite the term in the exponent as $n + m - 2k + 1 = n - m - 2(k - m) + 1 = n' - 2k' + 1$, where n' and k' are, respectively, the number of particles and the number of partitions that are left when disentangled m particles are left aside.

[16] First consider qubit systems and find an example of a state ρ and observables A, A'. Next consider a state $\rho \oplus \mathbf{0}$ and observables $A \oplus \mathbf{0}, A' \oplus \mathbf{0}$.

[17] What we have shown is that the relations (26) and (31) are entanglement witnesses. It is still open whether the relation (26) can be used to rule out hidden variable models.

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