The Feynman Propagator from a Single Path

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We show that it is possible to construct the Feynman propagator for a free particle in one dimension, without quantization, from a single continuous space-time path.

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The Feynman path-integral formulation of quantum mechanics [1,2] is well known for its utility and intuitive appeal. An interesting history of its development may be found in the article and book by Schweber [3,4]. Although the mathematics of the path integral encourages us to think of the paths in terms of real space-time trajectories, and there have been very interesting proposals for testing the reality of the paths [5–7], the formulation itself falls short of providing a full microscopic basis for quantum mechanics. This is in contrast to the Wiener integral which is an abstraction of the microscopic model (Brownian motion) supporting the diffusion equation. In particular, Wiener paths are known to approximate actual physical trajectories of diffusing particles, whereas the relation between Feynman paths and physical particles is not so direct.

There are two main barriers to an association between Feynman paths and any physical trajectory of a real particle. First of all there is a many-to-one correspondence between Feynman paths and the particle being described. Interference effects require this nonuniqueness since individual trajectories carry variable phase but not variable amplitude in the propagator [8]. Thus a physical particle cannot simply traverse a single Feynman path while propagating in space-time.

A second impediment is that the phase associated with Feynman paths is a wave concept grafted onto the particle paradigm. The physical origin of phase is unknown. Phase arises in conventional quantum mechanics through formal quantization and although the appearance of phase in the path-integral context is more subtle, there is nothing in the classical particle paradigm to motivate its presence.

In this Letter, we show that in the particular case of the Feynman chessboard model, one can modify the formulation so that the propagator in a space-time *region* can be constructed by a single continuous space-time curve. This is done by allowing particles to have trajectories with reversed time segments. Although this might seem conceptually "expensive," allowing this feature explicitly provides the physical mechanism which creates the phase of a wave function *without invoking an analytic continuation.* The propagator appears naturally as a pattern created by the (space-time) plane-filling path of a single point-particle. In the new formulation, the many-to-one aspect of Feynman paths is circumvented by sewing together an ensemble of chessboard paths into a single curve in such a way that formal quantization is unnecessary.

The chessboard or checkerboard model [2,9,10] extended Feynman's path-integral approach to the relativistic domain in order to incorporate electron spin. In this model, particles hop with speed $\pm c$ on a discrete space-time lattice with spacing ϵ. Choosing units in which $c = 1$, paths consist of diagonal segments resembling forward bishop's moves in chess [Fig. 1(a)].

A lattice approximation to the Kernel $K(b, a)$ for a particle to propagate from position a at time t_a to position b at time t_b is given by Feynman to be

$$K(b, a) = \sum_R N(R)(i\epsilon m)^R, \tag{1}$$

where the sum is over all chessboard paths and $N(R)$ is the number of paths with R corners. Here m is the mass of the particle in units where $\hbar$ is one. In terms of the paths themselves, the expected distance between corners is $1/m$ [10]. If we distinguish between the two directions in space, K is a 2×2 matrix which converges to the Dirac propagator in the continuum limit [9]. The prescription given in (1) can be modified somewhat for convenience. Gersch, who established the relation between the chessboard model and the one-dimensional Ising model, pointed out that the nonrelativistic limit is more direct if i is replaced by $-i$ in (1). Kull and Treumann [11] also noted that paths fixed at both ends have $(R - 1)$ degrees of freedom, so the R in (1) may be replaced by $(R - 1)$ without interfering with the continuum limit.

Equation (1) is a formal analytic continuation (quantization) of a classical partition function. The i in the sum, which replaces a real positive weight in the partition function, enforces the quantization. It also partitions the sum into four components, each of which is real, i.e.,

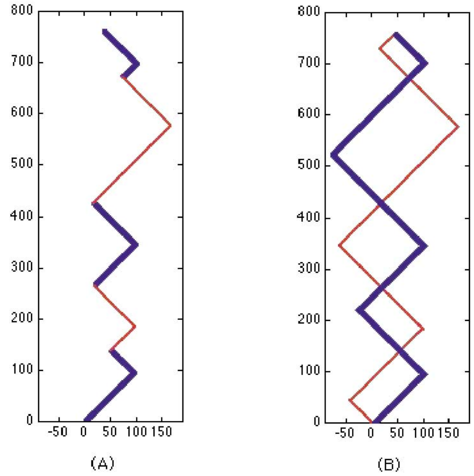

FIG. 1 (color online). (a) A Feynman chessboard trajectory. The x axis is horizontal and the t axis vertical. The sign of the contribution changes every two corners in the trajectory. This is indicated in the figure by the different linewidths in the different segments. (b) The same chessboard trajectory with its orthogonal twin. This pair can be viewed as two oscillating chessboard paths which never cross, or as a single entwined loop which crosses itself frequently. The latter view explains the phase shift of π for every two corners in the original chessboard paths.

$$K(b,a) = \left[\sum_{R=0,4,\ldots} N(R)(\epsilon m)^R - \sum_{R=2,6,\ldots} N(R)(\epsilon m)^R\right] + i\left[\sum_{R=1,5\ldots} N(R)(\epsilon m)^R - \sum_{R=3,7,\ldots} N(R)(\epsilon m)^R\right] = \Phi_R + i\,\Phi_I. \quad (2)$$

Each of the above sums is, by itself, a partition function for a class of random walks in which the term $(\epsilon m)^R$ is just a Boltzman weight. The interference of alternative paths is a result of the two subtractions in (2). If we replace the minus signs in (2) by plus signs, the resulting propagator is related to the telegraph equation, which in turn becomes the diffusion equation in the appropriate "nonrelativistic" limit [12], the remaining i then being superfluous. The underlying stochastic model for this case has been studied by Kac [13] and its relation to the Dirac equation through analytic continuation has been discussed by Gaveau *et al.* [14] and Jacobson and Schulman [10]. The i which appears in (2) just expresses K as a particularly convenient linear combination of the real amplitudes $\Phi_{R/I}$, however the actual interference characteristic of quantization is apparent in the oscillatory nature of the Φ.

Since it is the occurrence of the minus signs in the propagator which is essential for interference, we look for a physical basis for the subtractions. Regarding Fig. 1(a) we can encode the counting and subtractions involved in (2) by coloring the trajectories with two colors, say, blue (thick lines in figure) and red (thin in figure). If the trajectories start out blue, they change to red at the second corner, blue at the fourth, and so on. The sign of the contribution of a trajectory is then determined by its color at the space-time point in question, + for blue, − for red. Red contributions behave like antiparticles in that they reduce the contribution of the particles, providing interference effects. The ensemble of such colored paths between a and b provides the appropriate contribution to a quantum propagator, but is not explicitly traversed as a single path. What we would like to do is to sew together the chessboard paths in such a way that they may be traversed by a single path which also provides the alternating colors of the trajectories through the direction in time of the traversal. To this end let $t_R > t_b$ be a reversal time which will represent a rough upper bound in t for the generation of a propagator. We will use a small region in $t \geq t_R$ to join chessboard paths without appealing to the stochastic process. Now note from Fig. 1(b) that each chessboard path has an orthogonal twin.

The orthogonal twin starts from the origin moving in the opposite direction with the opposite color. It moves the same distance as the second leg of its twin's path, reverses direction, and moves the same distance as its twin's first leg. Twins meet at every second corner where they change both color and direction. For paths with an odd number of corners, this is repeated until the twins meet at $t = t_R$. (For paths with an even number of corners, see below.) The orthogonal twin is also a chessboard path with coloring 180° out of phase with the original.

Now consider the following "entwined" traversal of the two paths. Follow the first twin to the first meeting, the second to the second meeting, and so on. This path is blue from the origin to the last meeting. From there reverse the direction in t by proceeding down the remaining red sections. This brings you back to the origin on an entirely red path. This choice of traversal gives a meaning to the original Feynman coloring; the coloring corresponds to the direction in time of an *entwined* path traversal. Blue corresponds to forward in t, red to backwards. Entwined pairs also conserve charge if we associate opposite charges with reversed time segments.

Each chessboard trajectory to t_R has a unique orthogonal twin. Let P_R be an arbitrary n-step R-cornered chessboard path. Write $P_R = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ where $\sigma_k = \pm 1$ according to the direction of the kth step of the path. If we define a "leg" as a set of contiguous steps all in the same direction and bounded by either corners or ends of a path (i.e., a domain in the Ising analogy), then if R is odd, we may write $P_R = (l_1, l_2, \ldots, l_{R+1})$ with the understanding that l_1 stands for the first leg, l_2 stands for the second, and so on. If R is even then the path ends with the last link in the same direction as the first link. In order to join the path to an orthogonal twin we need to add a final leg in the opposite direction. There are many ways to do this.

For uniqueness we add a final leg the same length as the original last leg but in the opposite direction. (This modifies the distribution of the last corner of the orthogonal twin, but does not affect either twin below the last intersection before t_R. In terms of the resulting propagator, this results in an error which decays exponentially in $|t - t_R|$ and is small provided t_R is greater than a few Compton time steps from t.) If R is even, we thus extend the n-step path to $P_R = (l_1, l_2, \ldots, l_{R+1}, -l_{R+1})$, where $-l_{R+1}$ is l_{R+1} with the signs of all the component σ changed. We may then define the orthogonal twin to P_R as

$$P_R^\dagger = \begin{cases} (-l_1, l_1) & R = 0, \\ (l_2, l_1) & R = 1, \\ (l_2, l_1, \ldots, l_R, l_{R-1}, -l_{R+1}, l_{R+1}) & R = 2, 4, \ldots, \\ (l_2, l_1, l_4, l_3, \ldots, l_{R+1}, l_R) & R = 3, 5, \ldots. \end{cases} \tag{3}$$

Because $P_R^\dagger$ is a unique permutation of P_R, the ensemble, $\mathcal{E}_F$, of all extended n-step paths P_R from the origin is the same as the ensemble of all paths $P_R^\dagger$ from the origin. Furthermore, this is the same as the ensemble of paths of the form $(+1, \sigma_2, \ldots, \sigma_n)$ combined with all orthogonal twins. Thus we may cover all paths in $\mathcal{E}_F$, with the correct chessboard coloring, just by traversing all entwined pairs. This may be done through a single continuous (in the sense of the lattice) path, since all entwined loops intersect at the origin. Furthermore, entwined pairs fixed at the origin and at time t_R have the same number of degrees of freedom as their individual component chessboard paths (i.e., $R - 1$) and each pair may be given the statistical weight $(\epsilon m)^{R-1}$ which correctly weights the component chessboard trajectories to the last twin intersection before t_R. Thus the following classical stochastic process gives rise to a properly weighted chessboard ensemble of colored paths for a region of space-time $t < t_R$. Start a random walk at the origin and allow the walker to choose entwined paths out to t_R according to the number of free corners, either in the entwined path or one of the pairs. The walker traverses the entwined path as above so as to maintain both the chessboard and time-sense coloring. The walker ends up at the origin at the end of the traversal and repeats the process. The space-time lattice records the net number of traversals in the $+t$ direction as the walker passes by, registering a plus one for a positive traversal and a minus one for a negative traversal, thus accumulating positive and negative integers. Provided $t_R - t$ is greater than a few Compton lengths, the traversal weighting ensures that the constituent chessboard paths have the correct expected weight, and the ergodic nature of the walks ensures that, with enough loops, you can get as close as you like to a uniform coverage of the ensemble.

If we allow a walker to cycle through the entwined paths according to the above prescription, we can immediately write down the expected net "charge" accumulated on the lattice. Referring to the kernel in (1), we can define the four components of the 2×2 matrix as $K_{\sigma_n \sigma_1}$ where the subscripts refer to the end and beginning directions, respectively. If the walker, starting in the $+x$ direction, loops over N entwined pairs and (x, t) is a lattice point within the light cone with $t < t_R$ then the contribution to the $+x$-component of the net charge is proportional to $\rho_+ = N[K_{++}(x, t) - K_{+-}(x, t)]$. This is because an entwined loop corresponds to two forward chessboard paths, one originating from the origin with a positive, blue first leg (K_{++} contribution) and the second from a negative red first leg ($-K_{+-}$ contribution). Similarly the $-x$ component at (x, t) is proportional to $\rho_- = N[K_{-+}(x, t) - K_{--}(x, t)]$. These densities have the physical interpretation as particle densities which may be either positive or negative depending on the predominance of entwined trajectories in plus or minus t directions. ρ_+ is positive in $(+x, +t)$-rich areas and ρ_- is positive in $(-x, +t)$-rich areas. Note that $\rho_+ - \rho_-$ is proportional to the sum of the real and imaginary part of the Feynman propagator $\Phi_R + \Phi_I$ (Gersch convention for the sign of i).

Previous versions of this model [15–17] which used time-reversed segments to generate phase, showed that the paradigm worked in the continuum limit. However, it was only in the continuum limit that the Dirac propagator could be extracted without being swamped by the natural entropy of the stochastic process, the continuum limit also being a mean-field limit. Computer simulations on lattices then failed to extract the propagator unless stochastic fluctuations were completely surpressed, and the Feynman ensemble covered exactly. Using bound pairs in this model alleviates this problem and allows the propagator to be simulated on a lattice. Qualitatively, the bound pairs provide a microscopic "detailed balance" which ensures that statistical fluctuations in the coverage of the ensemble do not destroy macroscopic reversibility.

Figure 2 shows an example of a lattice simulation, where the sum of the real and imaginary parts of the propagator, $\Phi_R + \Phi_I$, at fixed t, is plotted versus x. The

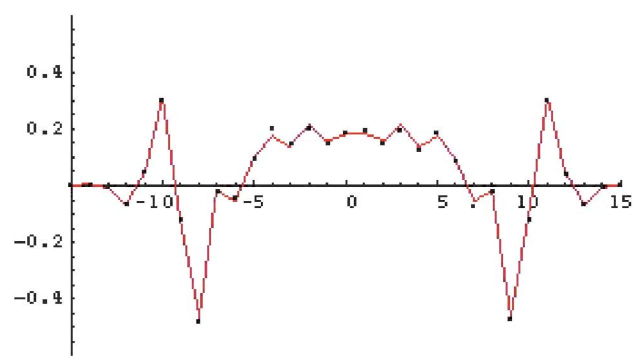

FIG. 2 (color online). The sum of propagator components, $\Phi_R + \Phi_I$ along the x axis, from the chessboard model (curve) and the single path simulation (points) at $t = 15$ steps from the origin.

expected results from the chessboard model are plotted (continuous curve) at the same lattice resolution as the results of a simulation with a single path which loops over the lattice 10^8 times. In the figure, t is 15 steps from the origin, with an average of two steps between corners or a probability of $1/2$ for a direction change at each step. At smaller values of t, this simulation and the chessboard model are indistinguishable on the scale of the figure, at larger values of t the single path gives sparser coverage of the chessboard ensemble and the scatter increases. The individual real and imaginary parts of the propagator may be calculated using the symmetry of the solutions, or by recording the $\rho_{\pm}$ in two components to separate contributions from the original chessboard path and its orthogonal twin.

Although we do not know how much of the above can survive inclusion of an external field and/or extension to three space dimensions, we do think the result reveals several qualitatively appealing features of the simplest case of a free particle in one dimension. First, the Feynman propagator has an independent existence as an expected net charge over an ensemble of entwined paths which can be joined into a single trajectory. In this context, the propagator has an underlying classical stochastic model which is in effect *self-quantizing* and produces real densities in place of amplitudes.

A second feature is that the above model provides a bridge between two distinct views of quantum mechanics in this case. Regarding Fig. 1(b), we may view the two trajectories in three ways. We can consider them as two separate chessboard trajectories, colored according to Feynman's corner rule. An ensemble of such trajectories builds a quantum propagator as a sum-over-histories. This is the conventional view.

A second picture is to note that an entwined pair forms a chain of creation/annihilation events. An ensemble of these would provide a vacuum of virtual particles upon which an excitation could presumably propagate. This is close to a field theory perspective.

The third picture, which is suggested by the new formulation, is the continuous loop in space-time, colored according to direction of motion in time. In this picture, the phase of the wave function, *Zitterbewegung*, and the presence of virtual particles are all manifestations of a single path which forms entwined space-time loops. In many respects, this picture is an implementation of the original Wheeler-Feynman one-electron-universe [4], scaled down to provide a single-path electron. Here the multiple tracks in space-time create a "Dirac sea" rather than the multitude of electrons in the universe.

To conclude, in demonstrating that entwined paths produce the retarded propagator *without analytic continuation*, we have reversed the usual relationship between geometry and quantization. In the entwined model, space-time geometry is fundamental and "quantization" is a derived feature. If this generalizes, it may prove useful in situations where conventional quantization is problematic.

An interesting next step, besides exploring generalizations, will be to define measurement in entwined systems in terms of localization of the single particle, the aim being to see if a traversal scheme could be found which would *imply* the Born postulate.

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