Replica Field Theories, Painlevé Transcendents, and Exact Correlation Functions

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Exact solvability is claimed for nonlinear replica σ models derived in the context of random matrix theories. Contrary to other approaches reported in the literature, the framework outlined does not rely on traditional "replica symmetry breaking" but rests on a previously unnoticed exact relation between replica partition functions and Painlevé transcendents. While expected to be applicable to matrix models of arbitrary symmetries, the method is used to treat fermionic replicas for the Gaussian unitary ensemble (GUE), chiral GUE (symmetry classes A and AIII in Cartan classification) and Ginibre's ensemble of complex non-Hermitian random matrices. Further applications are briefly discussed.

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Replica field theories are notoriously known for subtleties involved in performing the replica limit, $n \to 0$, devised [1] to carve a physical observable of interest out of the "annealed" average

$$Z_n(\epsilon_1, \ldots, \epsilon_p) = \left\langle \prod_{k=1}^{p} \det{}^n(\epsilon_k - \mathcal{H}) \right\rangle \quad (1)$$

often called the replica partition function; generically, $\mathrm{Im}\,\epsilon_k \neq 0$. The average $\langle \cdots \rangle$ runs over an ensemble of stochastic Hamiltonians $\mathcal{H}$ which, throughout this Letter, will be modeled by random matrices [2] of prescribed symmetries. Given (1), spectral properties of $\mathcal{H}$ can be obtained from the p-point Green's function $G(\epsilon_1, \ldots, \epsilon_p) = \langle \prod_{k=1}^{p} \mathrm{tr}(\epsilon_k - \mathcal{H})^{-1} \rangle$ for which the replica limit reads

$$G(\epsilon_1, \ldots, \epsilon_p) = \lim_{n\to 0} \frac{1}{n^p} \frac{\partial^p}{\partial \epsilon_1 \cdots \partial \epsilon_p} Z_n(\epsilon_1, \ldots, \epsilon_p). \quad (2)$$

Equation (2) assumes mutual commutativity of the following operations: disorder averaging, differentiation, the replica limit, and a thermodynamic limit, if necessary. Being an undoubtedly correct mathematical identity under the conditions of commutativity, the recipe (2) may become a pitfall [3,4] if applied unconsciously in the context of replica field theories.

A difficulty lurks behind a field theoretic prescription to compute the partition function (1). Sketchy, an original random system is substituted by its $|n|$ identical noninteracting copies, or replicas. Each copy, exemplified by a single determinant $\det(\epsilon - \mathcal{H})$, is represented by a functional integral over an auxiliary field which is either bosonic or fermionic by nature depending on the sign of n. Exponentiating a random Hamiltonian $\mathcal{H}$, such a representation facilitates a nonperturbative averaging over an ensemble of stochastic Hamiltonians in (1) and eventually results in effective field theories defined on either a compact [5] ($n > 0$) or a noncompact [6] ($n < 0$) manifold.

The point to make is this: since the number $|n|$ of functional field integrals involved is a positive integer, $|n| \in \mathbb{N}$, the validity of the resulting representation of the replica partition function Z_n is restricted to $|n| \in \mathbb{N}$, too. Unfortunately, this is not enough to perform the replica limit (2) determined by the behavior of Z_n in a close vicinity of $n = 0$. Therefore, a procedure of analytic continuation of Z_n away from $|n|$ positive integers is called for.

Kamenev-Mézard's prescription.—To keep the discussion concrete, let us reconsider a problem [7] of evaluation of the one-point Green's function $G(\epsilon)$ for random matrices with Gaussian distributed entries. In what follows, we assume the Hamiltonian $\mathcal{H}$ to be drawn from the Gaussian unitary ensemble (GUE_N) [2] specified by the probability density $P_N(\mathcal{H}) \propto \exp(-\mathrm{tr}\mathcal{H}^2)$ for an $N \times N$ complex Hermitian matrix $\mathcal{H}$ to occur; $N \in \mathbb{N}$. The Green's function $G(\epsilon)$ is determined by the replica limit $G(\epsilon) = \lim_{n\to 0} n^{-1} \partial Z_n(\epsilon)/\partial \epsilon$ with

$$Z_n(\epsilon) = \langle \det{}^n(\epsilon - \mathcal{H}) \rangle_{\mathcal{H} \in \mathrm{GUE}_N}, \qquad n \in \mathbb{C}. \quad (3)$$

Angular brackets $\langle \cdots \rangle$ stand for a matrix integral over the properly normalized measure [2] $P_N(\mathcal{H})\mathcal{D}\mathcal{H}$.

In the above definition of Z_n, the parameter n is allowed to be an arbitrary complex number $\mathbb{C}$. Therefore, if applied directly to (3), the replica limit would result [8] in a correct answer for $G(\epsilon)$. This is not so, however, if one maps the partition function Z_n onto a variant [9] of the replica σ model [5], $Z_n \mapsto \tilde{Z}_n$. Following the standard steps of fermionic mapping, one derives [9,10]

$$\tilde{Z}_n(\epsilon) = \langle \det{}^N(i\epsilon - \mathcal{Q}) \rangle_{\mathcal{Q} \in \mathrm{GUE}_n}, \qquad n \in \mathbb{N}. \quad (4)$$

Contrary to the starting point (3), the replica parameter n in (4) is now restricted to positive integers *by derivation.* As a result, any attempt to reconstruct the Green's function $G(\epsilon)$ out of $\tilde{Z}_n$ through the replica limit (2) will inevitably face the problem of analytic continuation of $\tilde{Z}_n$ away from $n \in \mathbb{N}$.

Despite numerous efforts and discussions throughout more than two decades, no mathematically satisfactory idea was brought in, even though there exists a recipe [9,11] to deal with the problem. Its detailed exposition can be found in Ref. [12]; below we only recollect the facts needed for further discussion.

(i) As a prerequisite, one attempts to unveil an explicit dependence of $\tilde{Z}_n$ on the replica index n which is only implicit in (4). To this end, the matrix integral (4) is evaluated *approximately* through a saddle point procedure which makes sense if the dimension N of the random matrix $\mathcal{H}$ is large enough. For not too large replica parameter $n \in \mathbb{N}$ (in particular, n should not scale with N), this yields [12] $\tilde{Z}_n^{(\mathrm{sp})}(\epsilon) \simeq \sum_{p=0}^{n} \mathrm{vol}(G_{n,p}) z_{n,p}(\epsilon)$, where $\mathrm{vol}(G_{n,p}) = \prod_{j=1}^{p}[\Gamma(j)/\Gamma(n-j+1)]$ is the volume of Grassmannian $G_{n,p} = \mathrm{U}(n)/\mathrm{U}(n-p) \times \mathrm{U}(p)$, and $z_{n,p}(\epsilon)$ is a known function (its explicit form is irrelevant to our discussion). The inner index p in the above sum $\sum_{p=0}^{n}(\cdots)$ counts a hierarchy of (nonequivalent) causal [12] saddle points contributing the integral (4) over Hermitian matrix $\mathcal{Q}$; inequivalence of the saddle points highlights a phenomenon of "replica symmetry breaking."

(ii) Further, one seeks a proper analytic continuation of $\tilde{Z}_n^{(\mathrm{sp})}$ away from $n \in \mathbb{N}$. The (most successful so far) procedure devised in Ref. [9] suggests extending the summation over p to infinity, $\sum_{p=0}^{n}(\cdots) \mapsto \sum_{p=0}^{\infty}(\cdots)$, as the group volumes $\mathrm{vol}(G_{n,p})$ vanish for $p \geq n+1$. Such a proposal suffers from two major drawbacks [12]: (a) for $n \notin \mathbb{N}$ the group volumes grow too fast with p for the sum $\sum_{p=0}^{\infty}(\cdots)$ to converge; (b) so extended to infinity the sum over p would necessarily involve a contribution from $n \sim \mathcal{O}(N)$ where the summand is no longer given by $\mathrm{vol}(G_{n,p}) z_{n,p}(\epsilon)$. This questions the self-consistency of the method as a whole making it somewhat deficient. Despite all these drawbacks, however, the approach furnishes [9] a correct result for the large-N GUE density of states in both leading and subleading orders in $1/N$.

Integrable hierarchies and exact Painlevé reduction.—On brief reflection, one has to admit that the *approximate* evaluation of $\tilde{Z}_n(\epsilon)$ is the key point to blame for inconsistencies encountered in the procedure of analytic continuation. For this reason, we opt a route based on *exact* and, therefore, truly nonperturbative evaluation of the replica partition function. As improbable or fantastic as it sounds, this is not an impossible task.

Our claim of exact solvability of the replica model (4) and the models of the same ilk rests on two observations.

(i) To make the first, we routinely reduce the average over $\mathcal{Q} \in \mathrm{GUE}_n$ in (4) to the n-fold integral [9]

$$\tilde{Z}_n(\epsilon) = \int_{-\infty}^{+\infty} \prod_{k=1}^{n} d\lambda_k e^{-\lambda_k^2} (\lambda_k - i\epsilon)^N \Delta_n^2(\lambda). \qquad (5)$$

Here, $\Delta_n(\lambda) = \prod_{k>\ell=1}^{n}(\lambda_k - \lambda_\ell) = \det(\lambda_k^{\ell-1})$ is the Vandermonde determinant [2] which makes it possible to bring (5) to the form [8]

$$\tilde{Z}_n(\epsilon) = e^{n\epsilon^2} \tilde{\tau}_n(\epsilon; N), \qquad n \in \mathbb{N}, \qquad (6)$$

which involves the Hänkel determinant $\tilde{\tau}_n(\epsilon; N) = \det[\partial_\epsilon^{k+\ell} \tilde{\tau}_1(\epsilon; N)]_{k,\ell=0,\ldots,n-1}$. The latter, as had first been shown by Darboux [13] a century ago, satisfies the equation [14]

$$\tilde{\tau}_n \tilde{\tau}_n'' - (\tilde{\tau}_n')^2 = \tilde{\tau}_{n-1} \tilde{\tau}_{n+1}, \qquad n \geq 1, \qquad (7)$$

given the initial conditions $\tilde{\tau}_0 \equiv 1$ and $\tilde{\tau}_1 = e^{-\epsilon^2} \tilde{Z}_1(\epsilon)$; the prime stands for $d/d\epsilon$. The structure of (7) is eventually due to the $\beta = 2$ symmetry of the replica field theory encoded into Δ_n^2 in (5); β is Dyson's index. Equations (6) and (7) establish a hierarchy between nonperturbative replica partition functions $\tilde{Z}_n$ with different $n \in \mathbb{N}$. This is an *exact* alternative to the *approximate* solution $\tilde{Z}_n^{(\mathrm{sp})}(\epsilon)$. *Equation* (7)*, known as the Toda lattice equation* [15] *in the theory of integrable hierarchies* [16]*, is the first indication of exact solvability hidden in replica field theories.*

(ii) The second observation borrowed from Ref. [17] concerns the fact that, miraculously, the same Toda lattice equation governs the behavior of so-called τ functions arising in the Hamiltonian formulation of the six Painlevé equations (PI–PVI), which are yet another fundamental object in the theory of nonlinear integrable systems. Complementary to (7), and also luckily, the Painlevé equations contain the hierarchy (or replica) index n as a *parameter*. For this reason, they serve as a proper starting point to build a consistent analytic continuation of nonperturbative replica partition functions away from n integers. *This Painlevé reduction confirms exact solvability of replica σ models and assists performing the replica limit* (2).

GUE density of states revisited.—With these observations in hand, let us consider the one-point Green's function for GUE_N where we have a rare luxury of examining the replica partition function as a function of energy ϵ, replica parameter n, and matrix dimension N both *before* and *after* replica σ model mapping.

After replica mapping, the Painlevé reduction of the partition function $\tilde{Z}_n(\epsilon)$ obeying (6) and (7) with $\tilde{Z}_0(\epsilon) \equiv 1$ and $\tilde{Z}_1(\epsilon) = H_N(\epsilon)$ [H_N is the Hermite polynomial; see (5) at $n = 1$] materializes in the exact representation [17]

$$\tilde{Z}_n(\epsilon) = \tilde{Z}_n(0) \exp\left(\int_0^{i\epsilon} dt \varphi_{\mathrm{IV}}(t)\right), \qquad n \in \mathbb{N}. \qquad (8)$$

It involves the Painlevé transcendent $\varphi_{\mathrm{IV}}(t) = \varphi_n(N; t)$ satisfying the Painlevé IV equation in the Jimbo-Miwa-Okamoto form [18]

$$(\varphi_{\mathrm{IV}}'')^2 - 4(t\varphi_{\mathrm{IV}}' - \varphi_{\mathrm{IV}})^2 + 4\varphi_{\mathrm{IV}}'(\varphi_{\mathrm{IV}}' - 2n)(\varphi_{\mathrm{IV}}' + 2N) = 0. \qquad (9)$$

The boundary condition is $\varphi_{\mathrm{IV}}(t) \sim (nN/t)(1 + \mathcal{O}(t^{-1}))$ as $t \to +\infty$. Note that (9), and therefore (8), contains the replica index n as a *parameter.*

By derivation, Eq. (8) holds for n positive integers only and, generically, there is no *a priori* reason to expect it to stay valid away from $n \in \mathbb{N}$. We claim, however, that it *is* legitimate to extend (8) and (9), as they stand, beyond $n \in \mathbb{N}$ and consider this extension as a sought analytic continuation. To prove this, we examine the partition function $Z_n(\epsilon)$ *prior* to σ model mapping as given by

(3). In the eigenvalue representation [2], Eq. (3) reduces to the N-fold integral akin to (5),

$$Z_n(\epsilon) = \int_{-\infty}^{+\infty} \prod_{k=1}^{N} d\lambda_k e^{-\lambda_k^2} (\lambda_k - \epsilon)^n \Delta_N^2(\lambda). \tag{10}$$

Similar to (5)–(7), the appearance of Δ_N^2 in (10) leads to the Hänkel determinant representation $Z_n(\epsilon) = Z_n(\epsilon; N) = e^{-N\epsilon^2}\tau_N(\epsilon; n)$ with $\tau_N(\epsilon; n) = \det[\partial_\epsilon^{k+\ell}\tau_1(\epsilon; n)]_{k,\ell=0,\ldots,N-1}$. Given the initial conditions $\tau_0 \equiv 1$ and $\tau_1 = e^{\epsilon^2} Z_n(\epsilon; 1)$, the Toda lattice equation $\tau_N \tau_N'' - (\tau_N')^2 = \tau_{N-1}\tau_{N+1}$, with $N \geq 1$, follows by the Darboux theorem [13]. Since n is allowed to be an arbitrary complex number in (3), the Toda equation determines a whole set of replica partition functions $Z_n(\epsilon)$ for *all* $n \in \mathbb{C}$. For $Z_n(\epsilon; 0) \equiv 1$ and $Z_n(\epsilon; 1) = H_n(i\epsilon)$ [see (10) at $N = 1$], the Painlevé reduction of the above Toda equation reads [17]

$$Z_n(\epsilon) = Z_n(0) \exp\left(\int_0^\epsilon dt \psi_{\rm IV}(t)\right), \qquad n \in \mathbb{C}. \tag{11}$$

Here, $\psi_{\rm IV}(t) = \psi_N(n; t)$ satisfies the Painlevé IV equation

$$(\psi_{\rm IV}'')^2 - 4(t\psi_{\rm IV}' - \psi_{\rm IV})^2 + 4\psi_{\rm IV}'(\psi_{\rm IV}' - 2N)(\psi_{\rm IV}' + 2n) = 0 \tag{12}$$

and matches the asymptotic behavior $\psi_{\rm IV}(t) \sim (nN/t) \times (1 + \mathcal{O}(t^{-1}))$ at infinity $t \to +\infty$.

A brief inspection reveals that (11) reduces to (8) because of the duality $i\varphi_n(N; it)|_{n\in\mathbb{N}} = \psi_N(n; t)$ that can easily be verified from (9) and (12) and from the boundary conditions at infinity. As the above duality between (9) and (12) formally holds beyond $n \in \mathbb{N}$, we conclude that (8) and (9) considered, as they stand, at arbitrary complex n furnish a proper analytic continuation. Then, it can be shown [19] that, in the large-N limit, the replica projection $G(\epsilon) = \lim_{n\to 0} n^{-1}\partial \tilde{Z}_n(\epsilon)/\partial\epsilon$ of the so-continued partition function (8) results in the correct expression $G(\epsilon) = \epsilon - \sqrt{\epsilon^2 - 2N}$ for the one-point Green's function. The famous Wigner's semicircle [2] for the level density $\nu_N(\epsilon) = -\pi^{-1}{\rm Im}G(\epsilon) = \pi^{-1}\sqrt{2N - \epsilon^2}$ readily follows.

Wigner-Dyson correlations in GUE.—The two-level correlation function in GUE (symmetry class A in Cartan classification) can be treated along the same lines. Defined in terms of the two-point Green's function $G(s = \epsilon_1 - \epsilon_2) = G(\epsilon_1, \epsilon_2)$ as $R(s) = (1/2)[{\rm Re}G(s) - 1]$, it can be obtained from the replica limit $G(s) = -\lim_{n\to 0} n^{-2}\partial^2 \tilde{Z}_n(s)/\partial s^2$; $\tilde{Z}_n(s)$ is the fermionic partition function. Taken at imaginary argument, it is given by the Verbaarschot-Zirnbauer integral [4]

$$\tilde{Z}_n(is) = \frac{e^{ns}}{s^{n^2}} \int_0^{2s} \prod_{k=1}^{n} d\lambda_k e^{-\lambda_k} \Delta_n^2(\lambda). \tag{13}$$

This is a Fredholm determinant [20] associated with a gap formation probability [2] within the interval $(2s, +\infty)$ in the spectrum of an auxiliary $n \times n$ Laguerre unitary ensemble. Utilizing the results of Refs. [20,21] we derive [19]

$$\tilde{Z}_n(is) = \exp\left(\int_0^{2s} dt \frac{\sigma_{\rm V}(t) - n^2 + nt/2}{t}\right). \tag{14}$$

Here, $\sigma_{\rm V}(t) = \sigma(n; t)$ satisfies the Jimbo-Miwa-Okamoto form of the Painlevé V equation [22]

$$(t\sigma_{\rm V}'')^2 - (\sigma_{\rm V} - t\sigma_{\rm V}')[\sigma_{\rm V} - t\sigma_{\rm V}' + 4\sigma_{\rm V}'(\sigma_{\rm V}' + n)] = 0 \tag{15}$$

with the boundary condition [19,23] $\sigma_{\rm V}(t) \sim n^2 e^{-t}/t$ as $t \to +\infty$.

The replica limit is governed by the behavior of $\tilde{Z}_n(s)$ in the vicinity of $n = 0$. In concert with the above discussion, we assume that (14) and (15) determine the desired analytic continuation. Expanding the solution to (15) around $n = 0$ yields [19] $\sigma_{\rm V}(t) = n^2 E_2(t) + \mathcal{O}(n^3)$, where $E_2(t)$ is the exponential integral $E_\ell(z) = \int_1^\infty dt e^{-zt} t^{-\ell}$, ${\rm Re}z > 0$. As a result, the small-n expansion of the partition function reads [19]

$$\ln\tilde{Z}_n(is) = ns - n^2[1 + \gamma + \ln(2s) + E_1(2s) - E_2(2s)] + \mathcal{O}(n^3). \tag{16}$$

Here, $\gamma = 0.577\ldots$ is the Euler constant.

To the best of our knowledge, this is the first nonperturbative evaluation of $\tilde{Z}_n$. Implementing the replica limit, one derives the two-point Green's function of the form $G(s) = 1 + 2is^{-2}\sin(s)e^{is}$. This, in turn, reproduces the celebrated Wigner-Dyson two-point correlation function [2] $R(s) = \pi\delta(s) - s^{-2}\sin^2(s)$. Note that this result holds for arbitrary s down to zero (compare to Ref. [9]).

Further examples.—The same strategy can be applied to demonstrate exact solvability of fermionic replica σ models for other random matrix ensembles associated with the Toda lattice hierarchy. Our partial list includes chiral GUE (chGUE) [24] (symmetry class AIII) and Ginibre's ensemble [25] of complex matrices with no further symmetries. A detailed account of the Painlevé reduction for these ensembles will be presented elsewhere [19]. Here we announce only small-n expansions of the corresponding nonperturbative replica partition functions. Adopting the notation of Ref. [26] for chGUE [their Eq. (12)] and of Ref. [27] for Ginibre's matrix model [their Eqs. (13) and (32)], we have derived [19] for the chGUE

$$\ln\tilde{Z}_\nu^{(n)}(\epsilon) = n\left(\nu\ln\epsilon + \int_0^\epsilon dtt[K_\nu(t)I_\nu(t) + K_{\nu-1}(t)I_{\nu+1}(t)]\right) + \mathcal{O}(n^2) \tag{17}$$

by the Painlevé III reduction (I_ν and K_ν are modified Bessel functions) while for Ginibre's complex matrices [28]

$$\ln\widetilde{Z}_n(z,\bar{z}) = n(z\bar{z})\left[1 + \frac{(z\bar{z})^N}{(N+1)!(N+1)}\,{}_2F_2(N+1,N;N+2,N+2;-z\bar{z})\right] + \mathcal{O}(n^2), \tag{18}$$

$$\ln\widetilde{Z}_n(z,\bar{z};\omega,\bar{\omega}) = n\left(2z\bar{z} + \frac{\omega\bar{\omega}}{2}\right) - n^2[1 + \gamma + \ln(\omega\bar{\omega}) + E_1(\omega\bar{\omega}) - E_2(\omega\bar{\omega})] + \mathcal{O}(n^3) \tag{19}$$

by the Painlevé V reduction (${}_2F_2$ is a hypergeometric function). Exact correlation functions for the above ensembles follow from (17)–(19) upon implementing proper replica limits. Out of three, the small-n expansion (17) is of particular interest as it holds for arbitrary values of the topological charge ν. An alternative calculational scheme [26] based on the saddle point evaluation of U(n) matrix integrals, much in line with [9], could reproduce exact results for ν half integers only, with exactness being secured [12] by the Duistermaat-Heckman theorem [29].

(i) The A, AIII, and Ginibre's random matrix models exhaust our list of ensembles illustrating exact solvability of fermionic replica field theories. The integrable structure of all of them (as well as of those for the matrix models from B, C, and D Cartan symmetry classes not considered in the Letter) is related to the Toda lattice (7) whose appearance is traced back to a particular $\beta = 2$ symmetry of the corresponding replica partition functions.

(ii) We expect the other random matrix ensembles exhibiting $\beta = 1$ and $\beta = 4$ symmetries (belonging to AI, BDI, CI and AII, CII, DIII classes in Cartan classification, respectively) to be exactly solvable as well. In those cases, integrable hierarchies related to the Pfaff lattice [30] are likely to arise.

(iii) Another application of the formalism developed would be getting further insight into controversies surrounding bosonic replicas which are known to be a total failure in the description [4] of spectral correlations in some ensembles while being quite successful in the description of others [27].

(iv) Finally, it would be desirable to figure out to what extent the nonperturbative Painlevé reduction of replica partition functions reported in this Letter is helpful in the replica treatment [31] of recently advocated universal "zero-dimensional" random Hamiltonians which include interactions [32].

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