Stabilizing and Tracking Unknown Steady States of Dynamical Systems

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An adaptive dynamic state feedback controller for stabilizing and tracking unknown steady states of dynamical systems is proposed. We prove that the steady state can never be stabilized if the system and controller in sum have an odd number of real positive eigenvalues. For two-dimensional systems, this topological limitation states that only an unstable focus or node can be stabilized with a stable controller, and stabilization of a saddle requires the presence of an unstable degree of freedom in a feedback loop. The use of the controller to stabilize and track saddle points (as well as unstable foci) is demonstrated both numerically and experimentally with an electrochemical Ni dissolution system.

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Control of dynamical systems is a classical subject in engineering science [1]. The revived interest of physicists in this subject started with an observation that chaotic motion can be converted to any of a large number of periodic motions via stabilization of unstable periodic orbits embedded in a chaotic attractor [2]. A number of new control techniques that use only small feedback perturbations and do not require a knowledge of the model equations have been developed in the field of controlling chaos. One of the most popular is the time-delayed feedback control method [3]. The method has been successfully implemented in quite diverse experimental contexts. However, Nakajima [4] proved a topological limitation that the method cannot stabilize torsion-free periodic orbits or, more precisely, orbits with an odd number of real positive Floquet exponents. To overcome this limitation, one of us has recently suggested the introduction of an unstable degree of freedom into a feedback loop [5].

Although the field of controlling chaos deals mainly with the stabilization of unstable periodic orbits, the problem of stabilizing unstable steady states of dynamical systems is of great importance for various technical applications. Stabilization of a fixed point by usual methods of classical control theory requires a knowledge of its location in the phase space. However, for many complex systems (e.g., chemical or biological), the location of the fixed points, as well as exact model equations, are unknown. In this case, adaptive control techniques capable of automatically locating the unknown steady state are preferable. An adaptive stabilization of a fixed point can be attained with the time-delayed feedback method [3,6,7]. However, the use of time-delayed signals in this problem is not necessary, and thus the difficulties related to an infinite dimensional phase space due to delay can be avoided. A simpler adaptive controller for stabilizing unknown steady states can be designed on a basis of ordinary differential equations (ODEs). The simplest example of such a controller utilizes a conventional low-pass filter described by one ODE. The filtered dc output signal of the system estimates the location of the fixed point, so that the difference between the actual and filtered output signals can be used as a control signal. An efficiency of such a simple controller has been demonstrated for different experimental systems [7]. Further examples involve methods which do not require knowledge of the position of the steady state but result in a nonzero control signal [8].

In this Letter, we introduce a generalized adaptive controller described by a system of ODEs and prove that it has a topological limitation concerning an odd number of real positive eigenvalues of the steady state. We show that the limitation can be overcome by implementing an unstable degree of freedom into a feedback loop. The feedback produces a robust method of stabilizing *a priori* unknown unstable steady states, saddles, foci, and nodes.

Simple example.—An adaptive controller based on the conventional low-pass filter, successfully used in several experiments [7], is not universal. This can be illustrated with a simple model:

$$\dot{x} = \lambda^s(x - x^\star) + k(w - x), \qquad \dot{w} = \lambda^c(w - x). \quad (1)$$

Here x is a scalar variable of an unstable one-dimensional dynamical system $\dot{x} = \lambda^s(x - x^\star)$, $\lambda^s > 0$ that we intend to stabilize. We imagine that the location of the fixed point $x^\star$ is unknown and use a feedback signal $k(w - x)$ for stabilization. The equation $\dot{w} = \lambda^c(w - x)$ for $\lambda^c < 0$ represents a conventional low-pass filter (rc circuit) with a time constant $\tau = -1/\lambda^c$. The fixed point of the closed loop system in the whole phase space of variables (x, w) is $(x^\star, x^\star)$ so that its projection on the x axes corresponds to the fixed point of the free system for any control gain k. If for some values of k the closed loop system is stable, the controller variable w converges to the steady state value $w^\star = x^\star$ and the feedback perturbation vanishes.

The closed loop system is stable if both eigenvalues of the characteristic equation $\lambda^2 - (\lambda^s + \lambda^c - k)\lambda + \lambda^s\lambda^c = 0$ are in the left half-plane $\mathrm{Re}\lambda < 0$. The stability conditions are $k > \lambda^s + \lambda^c$, $\lambda^s \lambda^c > 0$. We see immediately that the stabilization is not possible with a conventional low-pass filter since for any $\lambda^s > 0$, $\lambda^c < 0$, we have $\lambda^s \lambda^c < 0$ and the second stability criterion is not met. However, the stabilization can be attained via an unstable controller with a positive parameter λ^c. Electronically, such a controller can be devised as the rc circuit with a negative resistance. Figure 1 shows a mechanism of stabilization. For $k = 0$, the eigenvalues are λ^s and λ^c, which correspond to the free system and free controller, respectively. With the increase of k, they approach each other on the real axes, then collide at $k = k_1$ and pass to the complex plane. At $k = k_0$ they cross symmetrically into the left half-plane (Hopf bifurcation). At $k = k_2$ we have again a collision on the real axes and then one of the roots moves towards $-\infty$ and another approaches the origin. For $k > k_0$, the closed loop system is stable. An optimal value of the control gain is k_2 since it provides the fastest convergence to the fixed point.

Generalized adaptive controller.—Now we consider the problem of adaptive stabilization in general. Let

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{p}) \tag{2}$$

be the dynamical system with N-dimensional vector variable $\boldsymbol{x}$ and L-dimensional vector parameter $\boldsymbol{p}$ available for an external adjustment. Assume that an n-dimensional vector variable $\boldsymbol{y}(t) = \boldsymbol{g}(\boldsymbol{x}(t))$ [a function of dynamical variables $\boldsymbol{x}(t)$] represents the system output. Suppose that at $\boldsymbol{p} = \boldsymbol{p}_0$ the system has an unstable fixed point $\boldsymbol{x}^\star$ that satisfies $\boldsymbol{f}(\boldsymbol{x}^\star, \boldsymbol{p}_0) = 0$. The location of the fixed point $\boldsymbol{x}^\star$ is unknown. To stabilize the fixed point, we perturb the parameters by an adaptive feedback

$$\boldsymbol{p}(t) = \boldsymbol{p}_0 + kB[A\boldsymbol{w}(t) + C\boldsymbol{y}(t)], \tag{3}$$

where $\boldsymbol{w}$ is an M-dimensional dynamical variable of the controller that satisfies

$$\dot{\boldsymbol{w}}(t) = A\boldsymbol{w} + C\boldsymbol{y}. \tag{4}$$

Here A, B, and C are the matrices of dimensions $M \times M$, $M \times L$, and $n \times M$, respectively, and k is a scalar parameter that defines the feedback gain. The feedback is constructed in such a way that it does not change the steady state solutions of the free system. For any k, the fixed point of the closed loop system in the whole phase space of variables $\{\boldsymbol{x}, \boldsymbol{w}\}$ is $\{\boldsymbol{x}^\star, \boldsymbol{w}^\star\}$, where $\boldsymbol{x}^\star$ is the fixed point of the free system and $\boldsymbol{w}^\star$ is the corresponding steady state value of the controller variable. The latter satisfies a system of linear equations $A\boldsymbol{w}^\star = -C\boldsymbol{g}(\boldsymbol{x}^\star)$ that has a unique solution for any nonsingular matrix A. The feedback perturbation $kB\dot{\boldsymbol{w}}$ vanishes whenever the fixed point of the closed loop system is stabilized.

Small deviations $\delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}^\star$ and $\delta\boldsymbol{w} = \boldsymbol{w} - \boldsymbol{w}^\star$ from the fixed point are described by variational equations

$$\delta\dot{\boldsymbol{x}} = J\delta\boldsymbol{x} + kPB\delta\dot{\boldsymbol{w}}, \qquad \delta\dot{\boldsymbol{w}} = CG\delta\boldsymbol{x} + A\delta\boldsymbol{w}, \tag{5}$$

where $J = D_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{x}^\star, \boldsymbol{p}_0)$, $P = D_{\boldsymbol{p}}\boldsymbol{f}(\boldsymbol{x}^\star, \boldsymbol{p}_0)$, and $G = D_{\boldsymbol{x}}\boldsymbol{g}(\boldsymbol{x}^\star)$. Here $D_{\boldsymbol{x}}$ and $D_{\boldsymbol{p}}$ denote the vector derivatives (Jacobian matrices) with respect to the variables $\boldsymbol{x}$ and parameters $\boldsymbol{p}$, respectively. The characteristic equation for the closed loop system reads:

$$\Delta_k(\lambda) \equiv \begin{vmatrix} I\lambda - J & -k\lambda PB \\ -CG & I\lambda - A \end{vmatrix} = 0. \tag{6}$$

For $k = 0$ we have $\Delta_0(\lambda) = |I\lambda - J||I\lambda - A|$ and Eq. (6) splits into two independent equations $|I\lambda - J| = 0$ and $|I\lambda - A| = 0$ that define N eigenvalues of the free system $\lambda = \lambda_j^s$, $j = 1, \ldots, N$ and M eigenvalues of the free controller $\lambda = \lambda_j^c$, $j = 1, \ldots, M$, respectively. By assumption, at least one eigenvalue of the free system is in the right half-plane. The closed loop system is stabilized in an interval of the control gain k for which all eigenvalues of Eq. (6) are in the left half-plane $\mathrm{Re}\lambda < 0$.

The following theorem defines an important topological limitation of the above adaptive controller. It is similar to the Nakajima theorem [4] concerning the limitation of the time-delayed feedback controller.

Theorem.—Consider a fixed point $\boldsymbol{x}^\star$ of a dynamical system (2) characterized by Jacobian matrix J and an adaptive controller (4) with a nonsingular matrix A. If the total number of real positive eigenvalues of the matrices J and A is odd, then the closed loop system described by Eqs. (2)–(4) cannot be stabilized by any choice of matrices A, B, C, and control gain k.

Proof.—The stability of the closed loop system is determined by the roots of $\Delta_k(\lambda)$. Writing Eq. (6) for $k = 0$ in the basis where matrices J and A are diagonal, we have

$$\Delta_0(\lambda) = \prod_{j=1}^{N}(\lambda - \lambda_j^s)\prod_{m=1}^{M}(\lambda - \lambda_m^c). \tag{7}$$

Here λ_j^s and λ_m^c are the eigenvalues of the matrices J and A, respectively. Now from Eq. (6), we also have $\Delta_k(0) = \Delta_0(0)$, so Eq. (7) implies

$$\Delta_k(0) = \prod_{j=1}^{N}(-\lambda_j^s)\prod_{m=1}^{M}(-\lambda_m^c) \tag{8}$$

for all k. Since the total number of eigenvalues λ_j^s and λ_m^c

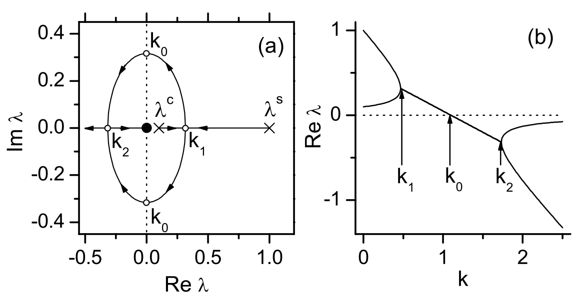

FIG. 1. Stabilizing an unstable fixed point with an unstable controller in a simple model of Eqs. (1) for $\lambda^s = 1$ and $\lambda^c = 0.1$. (a) Root loci of the characteristic equation as k varies from 0 to ∞. The crosses and solid dot denote the location of roots at $k = 0$ and $k \to \infty$, respectively. (b) $\mathrm{Re}\lambda$ vs k. $k_0 = \lambda^s + \lambda^c$, $k_{1,2} = \lambda^s + \lambda^c \mp 2\sqrt{\lambda^s\lambda^c}$.

that are real and positive is odd and other eigenvalues are real and negative or come in complex conjugate pairs, $\Delta_k(0)$ must be real and negative. On the other hand, from the definition of $\Delta_k(\lambda)$, we see immediately that when $\lambda \to \infty$ then $\Delta_k(\lambda) \to \lambda^{N+M} > 0$ for all k. $\Delta_k(\lambda)$ is an $N + M$ order polynomial with real coefficients and is continuous for all λ. Since $\Delta_k(\lambda)$ is negative for $\lambda = 0$ and is positive for large λ, it follows that $\Delta_k(\lambda) = 0$ for some real positive λ. Thus, the closed loop system always has at least one real positive eigenvalue and cannot be stabilized, Q.E.D. [9].

From this theorem it follows that any fixed point $\boldsymbol{x}^\star$ with an odd number of real positive eigenvalues cannot be stabilized with a stable controller. In other words, if the Jacobian J of a fixed point has an odd number of real positive eigenvalues, then it can be stabilized only with an unstable controller whose matrix A has an odd number (at least one) of real positive eigenvalues.

Controlling an electrochemical oscillator.—The use of an unstable degree of freedom in a feedback loop is now demonstrated with control in an electrodissolution process, the dissolution of nickel in sulfuric acid. The main features of this process can be qualitatively described with a model proposed by Haim *et al.* [10]. The dimensionless model together with the controller reads:

$$\dot{e} = i - (1 - \Theta)\left[\frac{C_h \exp(0.5e)}{1 + C_h \exp(e)} + a \exp(e)\right], \tag{9a}$$

$$\Gamma\dot{\Theta} = \frac{\exp(0.5e)(1 - \Theta)}{1 + C_h \exp(e)} - \frac{bC_h \exp(2e)\Theta}{C_h c + \exp(e)}, \tag{9b}$$

$$\dot{w} = \lambda^c(w - i). \tag{9c}$$

Here e is the dimensionless potential of the electrode and Θ is the surface coverage of NiO + NiOH. An observable is the current

$$i = (V_0 + \delta V - e)/R, \qquad \delta V = k(i - w), \tag{10}$$

where V_0 is the circuit potential and R is the series resistance of the cell. δV is the feedback perturbation applied to the circuit potential, k is the feedback gain. From Eqs. (10) it follows that $i = (V_0 - e - kw)/(R - k)$ and $\delta V = k(V_0 - e - wR)/(R - k)$. We see that the feedback perturbation is singular at $k = R$.

In a certain interval of the circuit potential V_0, a free ($\delta V = 0$) system has three coexisting fixed points: a stable node, a saddle, and an unstable focus [Fig. 2(a)]. Depending on the initial conditions, the trajectories are attracted either to the stable node or to the stable limit cycle that surrounds an unstable focus. As is seen from Figs. 2(b) and 2(c), the coexisting saddle and the unstable focus can be stabilized with the unstable ($\lambda^c > 0$) and stable ($\lambda^c < 0$) controller, respectively, if the control gain is in the interval $k_0 < k < R = 50$. Figure 2(d) shows the stability domains of these points in the (k, V_0) plane. If the value of the control gain is chosen close to $k = R$, the fixed points remain stable for all values of the potential V_0. This enables a tracking of the fixed points by fixing the control gain k and varying the potential V_0. In general, a tracking algorithm requires a continuous updating of the target state and the control gain. Our new method finds the position of the steady states automatically. The method is robust enough in the examples investigated to operate without a change in control gain. We also note that the stability of the saddle and focus points can be switched by a simple reversal of sign of the parameter λ^c.

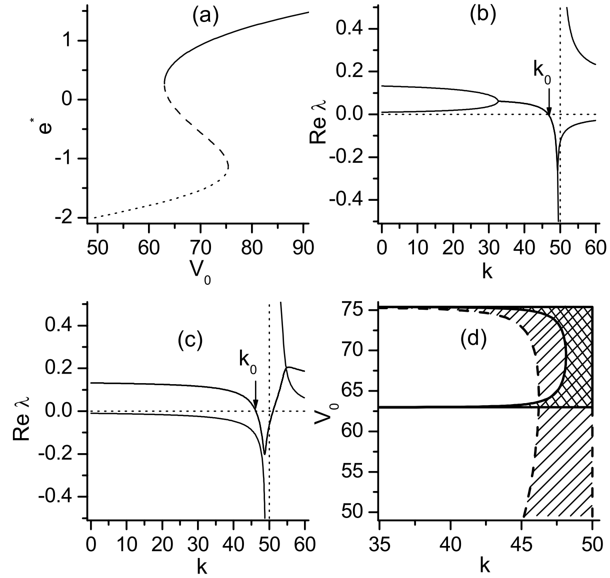

FIG. 2. Results of analysis of the electrochemical model for $R = 50$, $C_h = 1600$, $a = 0.3$, $b = 6 \times 10^{-5}$, $c = 10^{-3}$, $\Gamma = 0.01$. (a) Steady solutions $e^\star$ vs V_0 of the free ($\delta V = 0$) system. Solid, broken, and dotted curves correspond to a stable node, a saddle, and an unstable focus, respectively. (b),(c) Eigenvalues of the closed loop system as functions of control gain k at $V_0 = 63.888$ for the saddle $(e^\star, \Theta^\star) = (0, 0.0166)$ controlled by an unstable controller ($\lambda^c = 0.01$) and for the unstable focus $(e^\star, \Theta^\star) = (-1.7074, 0.4521)$ controlled by a stable controller ($\lambda^c = -0.01$), respectively. (d) Stability domain in (k, V_0) plane for the saddle (crossed lines) at $\lambda^c = 0.01$ and for the focus (inclined lines) at $\lambda^c = -0.01$.

Laboratory experiments have been carried out with nickel dissolution to verify the applicability of the proposed controller. A standard electrochemical cell consisting of a nickel working electrode (1 mm diameter), a $Hg/Hg_2SO_4/K_2SO_4$ reference electrode, and a Pt mesh counterelectrode was used. The current of the electrode is measured with a zero resistance ammeter, and the potential of the electrode [determined by Eqs. (9c) and (10)] is controlled with a Keithley Adwin Pro online controller system connected to the potentiostat. The data acquisition and control frequency was 200 Hz, larger than the inherent frequency of the oscillations (< 1 Hz).

The experimental parameters (concentration of sulfuric acid: 4.5 M, added external resistance 602 Ω, circuit potential V_0) have been optimized to show similar

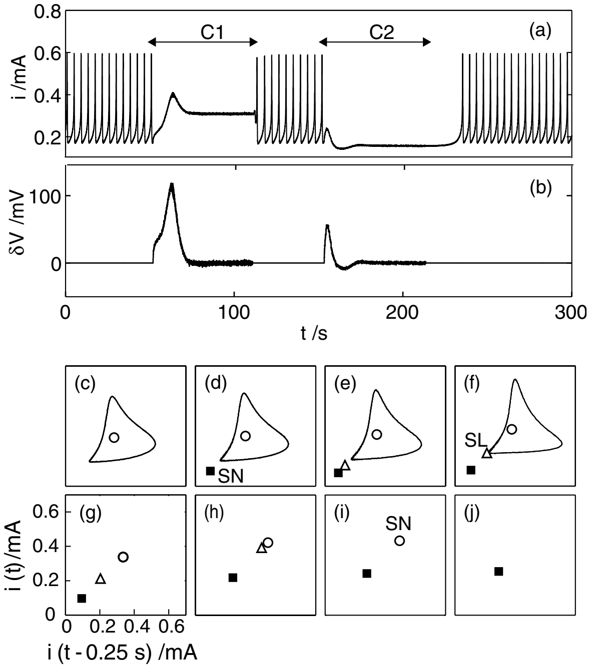

FIG. 3. Experiments. (a),(b) Controlling the unstable focus (region C1, $\lambda^c = -0.1\ \mathrm{s}^{-1}$) and the saddle point (region C2, $\lambda^c = 0.1\ \mathrm{s}^{-1}$) by changing the sign of λ^c. $k = 800\ \Omega$, $V_0 = 1.220$ V. (c)-(j) Phase portraits of different steady states (circle: high current unstable state, square: low-current stable state, triangle: saddle) and the limit cycles (solid lines) at different potentials V_0: (c) 1.200, (d) 1.210, (e) 1.220, (f) 1.260, (g) 1.270, (h) 1.380, (i) 1.400, and (j) 1.410 V. SN: saddle node bifurcation. SL: saddle-loop bifurcation.

dynamics to those of the simulations. At $V_0 = 1.240$ V periodic oscillations and a low-current, stable steady state are seen. (At a higher potential, about $V_0 = 1.270$ V, the oscillations disappear with finite amplitude and infinite period characteristic of a saddle-loop bifurcation.) In this parameter region the model predicts the existence of two additional unstable steady states: a high current unstable focus inside the limit cycle and a saddle point between the lower stable and the unstable higher one.

The stabilization of these latter two steady states can be achieved by implementing the control formula Eqs. (9c) and (10). The successful control is shown in Figs. 3(a) and 3(b) by keeping the feedback gain $k = 800\ \Omega$ and simply switching λ^c from $-0.1\ \mathrm{s}^{-1}$ to $0.1\ \mathrm{s}^{-1}$. The stable controller (region C1) stabilizes the higher current steady state with a vanishing control signal after a short transition period. The saddle point can be stabilized (region C2) with the unstable controller. The robustness of the control algorithm enabled us to stabilize unstable steady states in the whole parameter region of interest. By mapping the stable and unstable phase objects, we managed to visualize bifurcations from experimental data. In Figs. 3(c)–3(j) the stable steady states and limit cycles are shown with the stabilized unstable states. At lower potentials [Fig. 3(c)] there is only a stable limit cycle and an unstable focus. As the potential is increased [Fig. 3(d)] a new steady state occurs in the low-current region via a saddle-node bifurcation. At a somewhat larger potential [Fig. 3(e)] the resolution of a distinct saddle point and a stable node becomes possible. With further increase in the potential [Fig. 3(f)] the saddle point approaches the limit cycle; the limit cycle disappears with the collision of the saddle point [Fig. 3(g)], resulting in a saddle-loop (homoclinic) bifurcation. At larger potentials the saddle approaches the upper steady state and disappears through another saddle-node bifurcation. At large potentials [Fig. 3(j)] only one stable steady state exists.

In conclusion, we have proposed an adaptive controller for stabilizing and tracking unknown steady states of dynamical systems and demonstrated its efficiency with the chemical experiment. The controller is described by a finite set of ODEs and is simpler than a time-delayed feedback controller. We have shown that the adaptive stabilization of saddle-type steady states requires the presence of an unstable degree of freedom in a feedback loop.

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