CPT Violation Implies Violation of Lorentz Invariance

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(Received 28 January 2002; published 18 November 2002)

A interacting theory that violates *CPT* invariance necessarily violates Lorentz invariance. On the other hand, *CPT* invariance is not sufficient for out-of-cone Lorentz invariance. Theories that violate *CPT* by having different particle and antiparticle masses must be nonlocal.

DOI: 10.1103/PhysRevLett.89.231602 PACS numbers: 11.30.Er, 11.10.Lm, 11.30.Cp

A quantum field theory is Lorentz covariant in cone [1] if vacuum matrix elements of unordered products of fields (Wightman functions) are covariant. (What we call "functions" are distributions in a space of "generalized functions.") We assume in-cone Lorentz covariance (actually Poincaré covariance) in this paper. A quantum field theory is covariant out of cone if vacuum matrix elements of time-ordered products (τ functions) are covariant. To calculate the S matrix we need τ functions, or similar functions, such as retarded or advanced products (r functions or a functions). We require covariance of a quantum field theory both in and out of cone as the condition for Lorentz invariance of the theory; thus both the Wightman functions and the τ (or r or a) functions must be covariant for the theory to be Lorentz invariant.

Jost proved the fundamental theorem [2] that weak local commutativity at Jost points is necessary and sufficient for *CPT* symmetry. Jost points are spacetime points in which all convex combinations of the successive differences are spacelike, i.e., if the Wightman function is

$$W^{(n)}(x_1, x_2, \ldots, x_n) = \langle 0|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle, \quad (1)$$

a Jost point is an ordered set of $\{x_i\}$ such that all sums, $\sum_i c_i(x_i - x_{i+1})$, $c_i \geq 0$, $\sum_i c_i > 0$, are spacelike. Weak local commutativity states that

$$W^{(n)}(x_1, x_2, \ldots, x_n) = W^{(n)}(x_n, x_{n-1}, \ldots, x_1). \quad (2)$$

Clearly local commutativity implies weak local commutativity.

We will show that violation of *CPT* invariance in *any* Wightman function [3] implies noncovariance of the related τ (or r or a) function, and thus implies a violation of Lorentz invariance of the theory. We give our explicit discussion in terms of a scalar theory; however analogous arguments apply for any spin.

Kostelecký and collaborators have studied *CPT*- and Lorentz-violating theories systematically. Among their papers are [4,5]. The proceedings of a meeting on *CPT* and Lorentz symmetry are in [6]. Kostelecký *et al.* have emphasized the difference between observer (sometimes called "passive" in theories without Lorentz violation) Poincaré transformations in which coordinates are transformed according to $x \to \Lambda^{-1}(x - a)$, where Λ is a Lorentz transformation and a is a spacetime translation, and particle (sometimes called "active" in theories without Lorentz violation) Poincaré transformations in which fields are transformed according to

$$U(a, \Lambda)\phi(x)U(a, \Lambda)^{\dagger} = \phi(\Lambda x + a), \qquad U(a, \Lambda)|0\rangle = |0\rangle. \quad (3)$$

The analogous distinction exists for *CPT* symmetry. Unless stated otherwise, we always consider observer symmetries in this paper. If *CPT* is violated for any τ function, which implies that weak local commutativity is violated for that function, then the corresponding τ function is not Lorentz covariant (not Lorentz invariant for the scalar case we discuss explicitly), and the theory is not Lorentz invariant. A general τ function is given in terms of Wightman functions as

$$\tau^{(n)}(x_1, x_2, \ldots, x_n) = \sum_P \theta(x^0_{P_1}, x^0_{P_2}, \ldots, x^0_{P_n}) W^{(n)}(x_{P_1}, x_{P_2}, \ldots, x_{P_n}), \quad (4)$$

where θ enforces $x^0_{P_1} \geq x^0_{P_2} \geq \cdots \geq x^0_{P_n}$. In order for the τ function to be Lorentz covariant, Lorentz transformations that reverse the time order of points while leaving the Wightman function invariant must not change the τ function. This requires the equality of Wightman functions of permuted field orders when the relative distances are spacelike. If the functions are at a Jost point and the $\{x_i\}$ are such that all the successive time differences are positive, then there is an observer Lorentz transformation that leaves the Wightman functions invariant, but makes all the successive time differences negative. Invariance of the τ function requires that the original Wightman function and the one with the fields in completely reversed order have the same value. This is precisely the condition of weak local commutativity which is necessary and sufficient for *CPT* invariance of the corresponding matrix element. Thus if *CPT* invariance does not hold for this matrix element, then the τ function is not Lorentz invariant and the theory is not Lorentz invariant. This argument does not apply to a noninteracting theory for which τ functions need not be considered. Thus we have demonstrated the main result of this paper. *If CPT* *invariance is violated in an interacting quantum field theory, then that theory also violates Lorentz invariance.*

On the other hand, weak local commutativity insures only one of the equalities among Wightman functions of fields in permuted orders that are necessary for out-of-cone Lorentz covariance of the τ functions, so CPT invariance is not sufficient for out-of-cone Lorentz invariance.

We remark that weak local commutativity of Wightman functions implies weak local commutativity of truncated Wightman functions. We see no reason why violations of weak local commutativity cannot occur independently in truncated Wightman functions of different order. Even if the two-point Wightman function obeys weak local commutativity (which is the same condition as local commutativity for this case) which implies equal masses for the particle and antiparticle, weak local commutativity can be violated for the higher Wightman functions, and thus CPT can be violated in scattering and other physical processes even when the masses of particle and antiparticle are equal.

These results are relevant to theories in which the effective four-dimensional theory comes from a higher dimensional theory: if the effective four-dimensional theory violates CPT symmetry it also violates Lorentz invariance.

We consider the case in which the particle and antiparticle have different masses [7–9] specifically. We disagree with the assertion [8,9] that such a theory can be Lorentz covariant. We discuss the case of a charged scalar field; the results for other spin fields are qualitatively the same. We use covariant normalization for the annihilation and creation operators. We take the (Bose) commutation relations for the asymptotic (in or out) particles and antiparticles (we drop the labels in or out to simplify notation) for any observer to be

$$[a(\mathbf{p}), a^\dagger(\mathbf{p}')] = 2E(\mathbf{p})\delta(\mathbf{p}-\mathbf{p}'), \qquad E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}, \tag{5}$$

$$[b(\mathbf{p}), b^\dagger(\mathbf{p}')] = 2\bar{E}(\mathbf{p})\delta(\mathbf{p}-\mathbf{p}'), \qquad \bar{E}(\mathbf{p}) = \sqrt{\mathbf{p}^2 + \bar{m}^2}. \tag{6}$$

For $m^2 \neq \bar{m}^2$ these commutation relations violate C and CPT, but not necessarily P and T. These relations do not seem to occur in the general analysis of Kostelecký *et al.* [4–6]; we believe this is because Kostelecký *et al.* assume observer Lorentz covariance both in and out of cone, while we assume only in-cone observer Lorentz covariance.

We choose the Hamiltonian for this case to be

$$H = \int\left[\frac{d^3p}{2E(\mathbf{p})}E(\mathbf{p})a^\dagger(\mathbf{p})a(\mathbf{p}) + \frac{d^3p}{2\bar{E}(\mathbf{p})}\bar{E}(\mathbf{p})b^\dagger(\mathbf{p})b(\mathbf{p})\right]. \tag{7}$$

The usual commutation relations of the Lie algebra of the Poincaré group,

$$[P^\mu, P^\nu] = 0, \qquad [M^{\mu\nu}, P^\lambda] = i(\eta^{\nu\lambda}P^\mu - \eta^{\mu\lambda}P^\nu), \tag{8}$$

$$[M^{\mu\nu}, M^{\alpha\beta}] = i(\eta^{\mu\beta}M^{\nu\alpha} - \eta^{\mu\alpha}M^{\nu\beta} + \eta^{\nu\alpha}M^{\mu\beta} - \eta^{\nu\beta}M^{\mu\alpha}) \tag{9}$$

are satisfied by the replacements

$$P^\mu \to p^\mu, \qquad M^{\mu\nu} \to i\left(p^\mu\frac{\partial}{\partial p_\nu} - p^\nu\frac{\partial}{\partial p_\mu}\right). \tag{10}$$

Carrying this over to the field operators gives Hermitian operators for the generators of the Lie algebra,

$$P^\mu = \int\left[\frac{d^3p}{2E(\mathbf{p})}a^\dagger(\mathbf{p})p^\mu a(\mathbf{p}) + \frac{d^3p}{2\bar{E}(\mathbf{p})}b^\dagger(\mathbf{p})\bar{p}^\mu b(\mathbf{p})\right], \tag{11}$$

$$M^{\mu\nu} = \int\left[\frac{d^3p}{2E(\mathbf{p})}a^\dagger(\mathbf{p})i\left(p^\mu\frac{\partial}{\partial p_\nu} - p^\nu\frac{\partial}{\partial p_\mu}\right)a(\mathbf{p}) + \frac{d^3p}{2\bar{E}(\mathbf{p})}b^\dagger(\mathbf{p})i\left(\bar{p}^\mu\frac{\partial}{\partial \bar{p}_\nu} - \bar{p}^\nu\frac{\partial}{\partial \bar{p}_\mu}\right)b(\mathbf{p})\right]. \tag{12}$$

Thus the free fields carry a representation of the Poincaré algebra. In particular, the Hamiltonian, $H = P^0$, generates time translations. As we will show below, the problem with Lorentz invariance only occurs out of cone and shows up explicitly when there are interactions.

We construct the spacetime dependence of the fields by using the generators of translations,

$$e^{iP\cdot x}a(\mathbf{p})e^{-iP\cdot x} = e^{-ip\cdot x}a(\mathbf{p}). \tag{13}$$

Thus the x-space fields are

$$\phi(x) = \frac{1}{(2\pi)^{3/2}}\int\left[\frac{d^3p}{2E(\mathbf{p})}a(\mathbf{p})e^{-ip\cdot x} + \frac{d^3p}{2\bar{E}(\mathbf{p})}b^\dagger(\mathbf{p})e^{i\bar{p}\cdot x}\right], \tag{14}$$

$$\phi^\dagger(x) = \frac{1}{(2\pi)^{3/2}}\int\left[\frac{d^3p}{2\bar{E}(\mathbf{p})}b(\mathbf{p})e^{-i\bar{p}\cdot x} + \frac{d^3p}{2E(\mathbf{p})}a^\dagger(\mathbf{p})e^{ip\cdot x}\right], \tag{15}$$

where $p^0 = E(\mathbf{p})$ for terms with a and $a^\dagger$, and $\bar{p}^0 = \bar{E}(\mathbf{p})$ for terms with b and $b^\dagger$.

In this case the only truncated vacuum matrix elements are the two-point functions, which are covariant (invariant for this scalar case),

$$\langle 0|\phi(x)\phi^\dagger(y)|0\rangle = \Delta^{(+)}(x-y;m^2) = \frac{1}{(2\pi)^3}\int\frac{d^3p}{2E(\mathbf{p})}e^{-ip\cdot x}, \tag{16}$$

$$\langle 0|\phi^\dagger(x)\phi(y)|0\rangle = \Delta^{(+)}(x-y;\bar{m}^2) = \frac{1}{(2\pi)^3}\int\frac{d^3p}{2\bar{E}(\mathbf{p})}e^{-i\bar{p}\cdot x}. \tag{17}$$

The field is local in sense (iii) defined below if the commutator vanishes at a spacelike separation,

$$[\phi(x), \phi^\dagger(y)] = 0, \qquad (x-y)^2 < 0. \tag{18}$$

For this to hold the vacuum matrix element, $[\Delta^{(+)}(r; m^2) - \Delta^{(+)}(-r; \bar{m}^2)]$, $r^\mu = x^\mu - y^\mu$, must vanish at $r^2 < 0$. The asymptotic limit for $\sqrt{-r^2} \to \infty$ is

$$\langle 0|[\phi(x), \phi^\dagger(y)]|0\rangle \to (-r^2)^{-3/4}(e^{-m\sqrt{-r^2}} - e^{-\bar{m}\sqrt{-r^2}}); \tag{19}$$

this requires $m^2 = \bar{m}^2$.

Under Poincaré transformations the two-point τ function, which is the Feynman propagator,

$$\langle 0|T(\phi(x)\phi^\dagger(y))|0\rangle = \theta(x^0 - y^0)\langle 0|\phi(x)\phi^\dagger(y)|0\rangle + \theta(y^0 - x^0)\langle 0|\phi^\dagger(y)\phi(x)|0\rangle \tag{20}$$

becomes

$$\langle 0|T(\phi(\Lambda^{-1}(x-a))\phi^\dagger(\Lambda^{-1}(y-a)))|0\rangle = \theta((\Lambda^{-1}(x-y))^0)\langle 0|\phi(x)\phi^\dagger(y)|0\rangle + \theta((\Lambda^{-1}(y-x))^0)\langle 0|\phi^\dagger(y)\phi(x)|0\rangle, \tag{21}$$

where we used the translation and Lorentz invariance properties of $\Delta^{(+)}(x-y; m^2)$ and of $\Delta^{(+)}(y-x; \bar{m}^2)$. If $x-y$ is spacelike a Lorentz transformation can transform a vector with $x^0 > y^0$ into one with $y^0 > x^0$, which changes the value of the propagator from $\Delta^{(+)}(x-y; m^2)$ to $\Delta^{(+)}(y-x; \bar{m}^2)$. Thus the propagator is not covariant unless the vacuum matrix element of the commutator, $\Delta^{(+)}(x-y; m^2) - \Delta^{(+)}(y-x; \bar{m}^2)$, vanishes at spacelike separation and, as shown above, this happens only if $m^2 = \bar{m}^2$ [10].

Straightforward calculation of the τ function in momentum space gives

$$[2i(E(\mathbf{p}) - p^0 - i\epsilon)E(\mathbf{p})]^{-1} + [2i(\bar{E}(\mathbf{p}) + p^0 - i\epsilon)\bar{E}(\mathbf{p})]^{-1}. \tag{22}$$

For $m^2 = \bar{m}^2$ this reduces to the invariant form $i/(p^2 - m^2 + i\epsilon)$ as it should.

To illustrate the effect of the noninvariance of this propagator as viewed by different observers, assume the propagator mediates a scalar s-channel process $k_1 + k_2 \to k_1' + k_2'$, $k_1^2 = k_1'^2 = m_1^2$, $k_2^2 = k_2'^2 = m_2^2$. If the propagator were Lorentz invariant an observer who saw the total momentum to be zero (call this the center-of-mass frame) would find the same result as an observer who saw the momentum of one of the particles, say, particle 2 to be zero (call this the lab frame). (Since the propagator in momentum space is just the Fourier transform of the propagator in position space, we know already that these results will not agree. The purpose of the following calculation is to show that in the high-energy limit the results disagree qualitatively.) In the center-of-mass frame, $\mathbf{p} = 0$, $s = (\sqrt{\mathbf{k}^2 + m_1^2} + \sqrt{\mathbf{k}^2 + m_2^2})^2$, $E(\mathbf{p}) = m$, $\bar{E}(\mathbf{p}) = \bar{m}$. The propagator is

$$\text{propagator}|_{\text{cm}} = \frac{m^2 + \bar{m}^2 + (\bar{m} - m)\sqrt{s}}{2im\bar{m}(m - \sqrt{s})(\bar{m} + \sqrt{s})}. \tag{23}$$

In the lab frame, $\mathbf{p} = \mathbf{k}_1$, $s = m_1^2 + m_2^2 + 2m_2\sqrt{k_1^2 + m_1^2}$, $E(\mathbf{p}) = \sqrt{\mathbf{k}_1^2 + m^2}$, $\bar{E}(\mathbf{p}) = m_2$. The propagator is given by Eq. (22), with

$$E(\mathbf{p}) = \frac{\sqrt{s^2 - 2(m_1^2 + m_2^2)s + (m_1^2 - m_2^2)^2 + 4m_2^2m^2}}{2m_2}, \tag{24}$$

$\bar{E}(\mathbf{p})$ is similar, but with m replaced by $\bar{m}$, and $p^0 = (s + m_2^2 - m_1^2)/2m_2$. In the limit $s \to \infty$,

$$\text{propagator}|_{\text{cm}} \to \frac{i(\bar{m} - m)}{2m\bar{m}\sqrt{s}}, \tag{25}$$

$$\text{propagator}|_{\text{lab}} \to \frac{i}{s}. \tag{26}$$

Thus the large-s behaviors of the amplitude differ qualitatively in the two different frames. If $m = \bar{m}$ both propagators go to i/s for large s. For a resonant amplitude with s near m^2 or $\bar{m}^2$, this noninvariance of the propagator will lead to noninvariance of the scattering cross section.

Next we discuss the question of nonlocality when the masses of the particle and antiparticle differ.

Nonlocal quantum field theories were discussed extensively in the 1950's as a possible way to remove the ultraviolet divergences of quantum field theories [11–15]. The nonlocality was always introduced in the interaction terms, not in the quadratic terms that correspond to the free Hamiltonian. Nonlocal theories failed as a mechanism to solve the divergence problem. Surprisingly, some of these authors derived conditions under which the acausality due to nonlocality is restricted.

The property of locality can have three different meanings for a quantum field theory: (i) the fields enter terms in the Hamiltonian and the Lagrangian at the same spacetime point, (ii) the observables commute at spacelike separation, and (iii) the fields commute (for integer spin fields) or anticommute (for odd half-integer spin fields) at spacelike separation. Theories in which (i) fails can still obey (ii) and (iii), for example, quantum electrodynamics in the Coulomb gauge. Theories in which (iii) fails can still obey (i) and (ii), for example, parastatistics theories of order greater than 1. We have already shown above that the theory in which CPT is violated due to having different masses for the particles and antiparticles is nonlocal in sense (iii). We expect that such a theory will be nonlocal in sense (ii), but we do not show this here. Next we do show that such a theory is nonlocal in sense (i).

We discuss the case of a charged scalar field explicitly; the results for other spin fields are qualitatively the same. We use the annihilation and creation operators given in

Eqs. (5) and (6). We calculate the free Hamiltonian for such fields in two ways. First, we calculate the Hamiltonian assuming only first derivatives of the fields enter and, secondly, we do the calculation allowing higher derivatives.

With only first order derivatives, we find

$$a(\mathbf{p}) = \frac{4E_p\bar{E}_p}{(E_p+\bar{E}_p)^2}\Big[\phi_p + \frac{E_p-\bar{E}_p}{2\bar{E}_p}\phi_{(-\bar{E}_p,\mathbf{p})} \times \exp(i(E_p+\bar{E}_p)x^0)\Big], \quad (27)$$

$$b(\bar{\mathbf{p}}) = -\frac{4E_p\bar{E}_p}{(E_p+\bar{E}_p)^2}\Big[\phi_{(-\bar{E}_p,\mathbf{p})} - \frac{E_p-\bar{E}_p}{2E_p}\phi_p \times \exp(-i(E_p+\bar{E}_p)x^0)\Big]. \quad (28)$$

Here we used the definition

$$\phi_p = \frac{i}{(2\pi)^{3/2}}\int d^3x e^{ip\cdot x}\overleftrightarrow{\partial}_{x^0}\phi(x). \quad (29)$$

The explicit nonlocal form of the Hamiltonian,

$$H = \frac{1}{2}\int d^3p[a^\dagger(\mathbf{p})a(\mathbf{p}) + b^\dagger(\bar{\mathbf{p}})b(\bar{\mathbf{p}})] \quad (30)$$

expressed in terms of $\phi(x)$ and $\phi^\dagger(x)$ follows from Eqs. (27)–(29); it is nonlocal in space, complicated, and not informative.

Allowing higher derivatives, however, leads to a relatively simple form for the Hamiltonian with the $\Delta^{(+)}$ function as the kernel that gives the nonlocality. The result is

$$H = \frac{2i}{(2\pi)^3(m^2-\bar{m}^2)^2}\Big\{\int d^3xd^3x'\frac{\partial\Delta^{(+)}(x-x';m^2)}{\partial(x^0-x'^0)}\frac{\overleftrightarrow{\partial}}{\partial x^0}\frac{\overleftrightarrow{\partial}}{\partial x'^0}(\partial_x\cdot\partial^x+\bar{m}^2)(\partial_{x'}\cdot\partial^{x'}+\bar{m}^2)\phi^\dagger(x)\phi(x') + \int d^3xd^3x'\frac{\partial\Delta^{(+)}(x-x';\bar{m}^2)}{\partial(x^0-x'^0)}\frac{\overleftrightarrow{\partial}}{\partial x^0}\frac{\overleftrightarrow{\partial}}{\partial x'^0}(\partial_x\cdot\partial^x+m^2)(\partial_{x'}\cdot\partial^{x'}+m^2)\phi(x)\phi^\dagger(x')\Big\}. \quad (31)$$

The apparent singularity in Eq. (31) due to the factor $(m^2-\bar{m}^2)^{-2}$ is removed by the Klein-Gordon operators and the Klein-Gordon scalar products in this equation.

To summarize, we have demonstrated that *CPT* invariance is necessary, but not sufficient, for Lorentz invariance of an interacting quantum field theory. We noted that violations of *CPT* can occur independently in different truncated Wightman functions.

We also showed that if one explicitly chooses different masses for particles and antiparticles the theory must be nonlocal in terms of the x-space fields associated with the particles. In that case the propagator is not covariant, and, further, the lack of covariance leads to qualitatively different behaviors of the propagator at large s in different frames of reference.

We are happy to thank Rabi Mohapatra for suggesting that we consider the relation between *CPT* and Lorentz invariance. We acknowledge helpful discussions with E. Boldt, T. Cohen, P. Frampton, T. Jacobson, W. Kummer, R. Mohapatra, S. Nussinov, and A. Shimony.

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[1] By "in cone" we mean that the momenta that are conjugate to the successive spacetime difference variables are physical, i.e., are in or on the forward light cone in momentum space. Similarly, "out of cone" means that unphysical momenta that can be outside the forward light cone are present (possibly) in addition to physical momenta.
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