

Measuring Neutron-Star Radii with Gravitational-Wave Detectors

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(Received 23 April 2002; published 18 November 2002)

Coalescing binary neutron stars (NS) are expected to be an important source of gravitational waves (GW) detectable by laser interferometers. We present here a simple method for determining the compactness ratio M/R of NS based on the observed deviation of the GW energy spectrum from point-mass behavior at the end of inspiral. Our method is based on the properties of quasiequilibrium binary NS sequences and does not require the computation of the full GW signal $h(t)$. Combined with the measurement of the NS masses during inspiral, the determination of M/R will allow very strong constraints to be placed on the equation of state of dense nuclear matter.

DOI: 10.1103/PhysRevLett.89.231102

PACS numbers: 04.30.Db, 47.11.+j, 95.85.Sz, 97.60.Jd

Coalescing compact binaries containing two neutron stars (NS) are among the most important sources of gravitational waves (GW) for LIGO [1], VIRGO [2], and other laser interferometers. Should the inspiral of such a binary be detected, the frequency evolution of the GW signal will immediately yield the system's "chirp mass" $M_{\text{ch}} \equiv \mu^{3/5} \mathcal{M}^{2/5}$, where μ and \mathcal{M} are the reduced and total mass of the binary, respectively. Higher-order post-Newtonian effects on the phase evolution of the signal also allow for the determination of the reduced mass μ , and thus the individual masses M_1 and M_2 of the two NS [3]. The determination of the NS radii in addition to their masses would yield important information about the equation of state (EOS) at nuclear densities, and could even indicate the presence of more exotic phases, such as strange quark matter instead of ordinary nuclear matter [4]. The GW signal of a coalescing binary could yield such information, but this is limited in two different ways. During the slow inspiral phase at large separations, i.e., low frequencies ($f \ll 1$ kHz), the stars behave like point masses, and finite-size effects are not expected to leave any signature in the GW signal [3,5,6]. During the final hydrodynamic merger, characteristic frequencies of GW emission could yield important information about the fluid EOS [7–9], but these frequencies are expected to lie well above the photon shot noise limit of current interferometers ($f \gtrsim 1$ kHz). Thus, it is only during the last few orbits of the inspiral, just prior to merger, that we can hope to see the imprint of the NS radii on a measurable GW signal (with $f \lesssim 1$ kHz).

Several groups have studied this terminal phase of inspiral by constructing quasiequilibrium sequences of close NS binaries in the conformal flatness approximation of general relativity (GR) [10–13]. In this approximation, it is assumed that the binary system evolves along a sequence of appropriately constructed equilibrium states with decreasing binary separation as energy is radiated away in GW. From the binary equilibrium energy curve

$E_{\text{eq}}(r)$, which gives the total system energy as a function of binary separation r , and the GW luminosity L_{GW} , one can derive the radial infall rate as $v_r = L_{\text{GW}}(dE_{\text{eq}}/dr)^{-1}$. With $v_r = dr/dt$ known, this provides the time evolution along the equilibrium sequence and the GW signal $h(t)$. This approach should remain accurate as long as the radial infall time scale r/v_r is longer than the dynamical time scale of the system, i.e., until the point where dynamical instability sets in, and the two stars plunge inward rapidly and merge.

Unfortunately, calculating the correct GW luminosity for a given matter configuration in GR is an extremely difficult task. Different approaches have required either time integration of the full nonlinear equations of GR [14] or the solution of a complicated wave equation for terms representing the spherical harmonic expansion of the GW metric perturbation [15]. The great complexity of these approaches is in stark contrast with the simplicity of the quasiequilibrium approximation. However, we point out here that the GW energy spectrum dE_{GW}/df can be calculated directly and very simply from the equilibrium energy curve, independent of any knowledge about the GW luminosity. Indeed, by definition of the quasiequilibrium approximation, the energy decrease $-dE_{\text{eq}}$ between two neighboring binary configurations along the sequence is equal to the energy dE_{GW} radiated away as the wave frequency sweeps up by df , where the GW frequency is twice the orbital frequency, $f = 2f_{\text{orb}}$. Thus, one should simply compute the total energy E_{eq} as a function of frequency f along the equilibrium sequence, and the GW energy spectrum is then given by the derivative $dE_{\text{GW}}/df = -dE_{\text{eq}}/df$. As a trivial example, consider the inspiral of two point masses in the Newtonian limit, where we have $E_{\text{eq}} \propto -r^{-1}$ and $f \propto r^{-3/2}$. It follows that $E_{\text{eq}} \propto -f^{2/3}$ and thus $dE_{\text{GW}}/df \propto f^{-1/3}$, a well-known result. In addition to the assumptions underlying the quasiequilibrium approximation, the validity of this simple approach relies on the additional

assumption that the GW emission during the later merger phase has no effect on the energy spectrum at lower frequencies. Indeed, this has been demonstrated in numerical hydrodynamic calculations of binary mergers [7,8], which show a clear separation between the inspiral and merger components of the emission in frequency space.

We have investigated the properties of the GW emission during the final phase of binary NS inspiral using new, highly accurate equilibrium sequences calculated with the LORENE code [16]. This code solves the five coupled, nonlinear, elliptic equations of the conformally flat approximation [10–13], using a multidomain spectral method [17,18]. This approach has already been used successfully in various astrophysical applications [13,19,20]. Typically, the computed fields satisfy the constraints of full GR to within $\sim 1\%$ [21], and LORENE configurations satisfy the virial theorem to within one part in 10^5 . The equilibrium sequences presented here are the natural extension of the work already published in [13,22]. They will be discussed in more detail in [23]. Here we show only the variation of the ADM (Arnowitt-Deser-Misner) mass of the system (total binary mass-energy $\mathcal{M} \equiv c^{-2}E_{\text{eq}}$) and the GW frequency (twice the orbital frequency), which are sufficient for our purposes to determine the GW energy spectrum.

The equilibrium configurations have been calculated for irrotational binaries, i.e., assuming that the fluid has zero vorticity in the inertial frame. Indeed, the NS should be spinning slowly at large separations and the viscosity of NS matter is too small for tidal spin-up to become significant on the coalescence time scale [5,24]. Based on the current set of well-measured NS masses in relativistic binary radio pulsars, it is expected that all NS in coalescing binaries will have masses $M \simeq 1.35M_{\odot}$ [25]. Hence, for simplicity, we consider only equal-mass binaries where $M_1 = M_2 = 1.35M_{\odot}$. Also for simplicity, we model the NS EOS with a simple polytropic form $P = K\rho^{\Gamma}$, where P is the pressure and ρ the rest-mass density. The constant K represents the overall compressibility of the matter and largely sets the value of the stellar radius for a given mass, while the adiabatic exponent Γ measures the stiffness of the EOS and the central concentration of the NS interior. Based on our experience with hydrodynamic calculations [8], we expect that the GW energy spectrum just prior to merger is determined primarily by the NS radius with relatively little sensitivity to Γ . For this reason, in this initial study, we allow the stellar radius R to vary for different NS models, but we set $\Gamma = 2$ for all models, as this value fits well most published NS EOS (see, e.g., [26] and references therein). Specifically, we consider NS models with compactness ratios $M/R = 0.12, 0.14, 0.16,$ and 0.18 (setting $G = c = 1$), where M is the ADM (gravitational) mass measured by an observer at infinity for a single isolated NS, and R is the circumferential radius of the NS. For $M = 1.35M_{\odot}$, the corresponding radii are $R = 16.6, 14.2, 12.4,$

and 11.1 km, respectively, spanning the range of values for NS radii calculated from various physical EOS. Note, however, that our results are unchanged under the rescaling given by $R' = \kappa R, M' = \kappa M, f' = f/\kappa$, for any constant κ .

For each NS model, about 12 binary equilibrium configurations are computed with decreasing separations, until a cusp develops on the NS surface. Equilibrium configurations for smaller separations do not exist. To each sequence we fit a curve of the form

$$\mathcal{M}(f) = 2.7M_{\odot} - k_{\text{N}}f^{2/3} + k_1f + k_2f^2 \quad (1)$$

to represent the variation of total mass energy as a function of GW frequency. The first term gives the total gravitational mass of the system at infinite separation, while the second term represents the Newtonian point-mass behavior, with $k_{\text{N}} = 2^{-4/3}\pi^{2/3}G^{2/3}M^{5/3} = 4.056 \times 10^{-4}M_{\odot} \text{ Hz}^{-2/3}$. The term $\propto f$ was introduced heuristically to represent the lowest-order post-Newtonian and finite-size corrections to the point-mass behavior at intermediate frequencies. The term $\propto f^2$ represents the tidal interaction energy, which causes the equilibrium energy curve to flatten at high frequencies. Our best (least-squares) fit values of k_1 and k_2 for each sequence are listed in Table I, and the results are illustrated in Fig. 1. The asterisks show the data points along each sequence, with a typical error between the data points and the fit of $\delta\mathcal{M} \sim 10^{-4}M_{\odot}$. We find in all cases that k_2 is positive, as we would expect: tidal deformations and relativistic gravitational effects increase the equilibrium energy [27,28]. We note that none of the equilibrium curves shows evidence of an energy minimum, which would have implied the onset of dynamical instability [6,22,27,29].

Computing the GW energy spectrum for each model now requires only differentiating the fitted curves with respect to frequency. The results are shown in Fig. 2. In each case, we see a characteristic frequency range where the spectrum plunges rapidly below the extrapolation of the low-frequency result. This corresponds to the flattening of the energy curves in response to the growing tidal interaction and post-Newtonian (PN) effects. Also shown is the energy spectrum of a 3PN, irrotational, point-mass binary, computed according to the results found in Ref. [30], which closely tracks our most compact model, indicating that the differences we see in the energy spectra result from finite-size effects associated with the NS radius. To quantify the deviations from the Newtonian case, we define a set of break frequencies, at which the energy spectrum has dropped by some factor below the point-mass result. The values we find for $f_{10}, f_{25},$ and f_{50} , where dE_{GW}/df has dropped by 10%, 25%, and 50%, respectively, are listed in the last three columns of Table I. We see that all these characteristic frequencies lie within the frequency range accessible by LIGO-type detectors, with perhaps the exception of f_{50} for the more compact sequences. Note that the calculation of f_{50} values requires extrapolating the equilibrium energy curves slightly

TABLE I. Properties of the quasiequilibrium sequences. Here M/R (with $G = c = 1$) is the compactness of an isolated NS seen by an observer at infinity, R is the circumferential radius for an ADM mass of $M = 1.35M_\odot$, f_c is the GW frequency at the final point of each sequence (cusp), k_1 and k_2 are the best fit parameters in Eq. (1), and f_{10} , f_{25} , and f_{50} are the break frequencies at which the GW energy spectrum has dropped, respectively, by 10%, 25%, and 50% below the point-mass energy spectrum.

M/R	R (km)	f_c (Hz)	k_1	k_2	f_{10} (Hz)	f_{25} (Hz)	f_{50} (Hz)
0.12	16.6	641	-4.939×10^{-6}	1.290×10^{-8}	342	518	764
0.14	14.2	807	-3.363×10^{-6}	9.244×10^{-9}	383	612	931
0.16	12.4	1002	-1.806×10^{-6}	6.490×10^{-9}	418	720	1137
0.18	11.1	1187	-5.834×10^{-7}	4.835×10^{-9}	431	810	1331

beyond the last equilibrium model (where a cusp develops), and may therefore be less reliable. However, we find that f_{50} has a particularly steep, quasilinear dependence on the NS compactness, given by $f_{50} \approx [10^4(M/R) - 460]$ Hz within the range of NS radii we considered. For comparison, f_{25} values can be determined safely within the quasiequilibrium approximation, and the sensitivity on compactness is only slightly reduced, with $f_{25} \approx [5000(M/R) - 85]$ Hz.

The best definition of the break frequency will be a trade-off between higher signal-to-noise ratio at lower

frequencies [31] and greater ease of discrimination between different EOS at the higher frequencies. In addition, the quasiequilibrium approximation is expected to be most accurate at lower frequencies, where the inspiral rate is slower. However, we doubt that this could become a major issue: if we adopt, for simplicity, the point-mass formula for the GW luminosity, and compute the

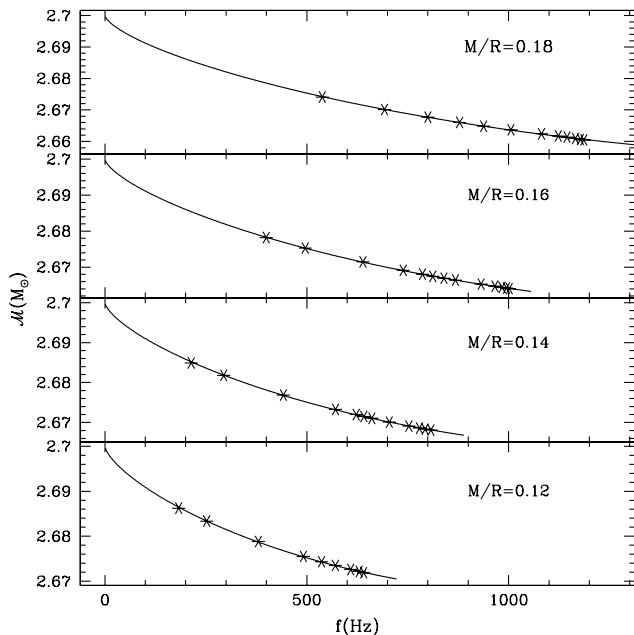


FIG. 1. ADM mass (total mass energy) of a binary NS system as a function of GW frequency (twice the orbital frequency), computed along each of our four irrotational equilibrium sequences. From bottom to top, the sequences correspond to NS with compactness $M/R = 0.12, 0.14, 0.16$, and 0.18 . All models assume a polytropic EOS with $\Gamma = 2$ and a NS mass of $1.35M_\odot$ for both components. The asterisks indicate the individual equilibrium configurations calculated along each sequence, while the lines show our best fit using Eq. (1) and the values of Table I.

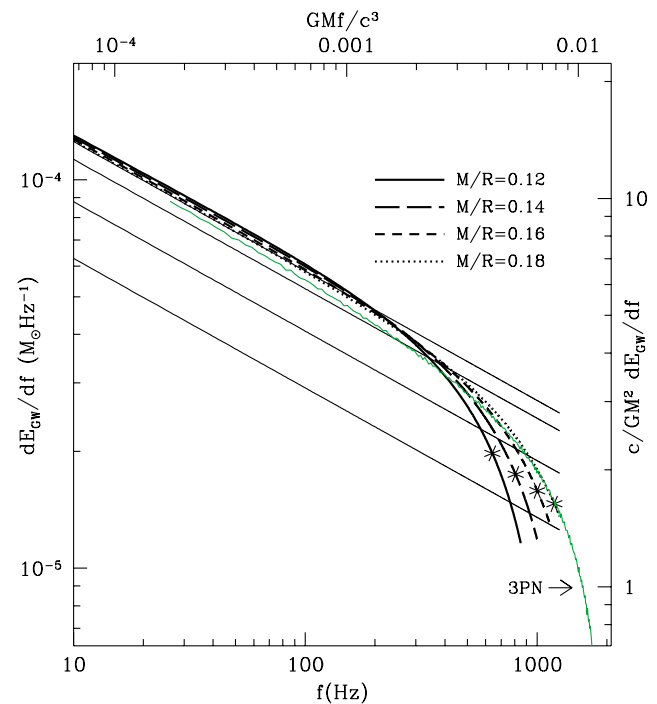


FIG. 2 (color online). Energy spectrum dE_{GW}/df of GW emission emitted along each of the four sequences of Fig. 1 in the quasiequilibrium approximation. Also shown is an irrotational 3PN point mass binary from Ref. [30], which closely tracks our most compact model. Asterisks indicate the terminal point along each sequence, where a cusp develops. The slanted straight lines show, from top to bottom, the point-mass Newtonian energy spectrum ($\propto f^{-1/3}$) multiplied by 1.0, 0.9, 0.75, and 0.5. The last three values are used to define characteristic break frequencies f_{10} , f_{25} , and f_{50} , where the energy spectrum has dropped by the corresponding fraction. The units on the right and top axes show the corresponding dimensionless quantities, with the mass dependence scaled away.

corresponding radial infall velocity along our equilibrium sequences, we find that v_r never exceeds 5% of the orbital velocity, even at the point where we define f_{50} (the corresponding fraction at the point where we define f_{25} is about 2%) [32]. Ultimately, the optimum choice should be determined by data analysts, taking into account the accuracy with which the break frequencies can be extracted using matched filtering or other techniques [3,33]. Preliminary studies of this problem have already been performed for both broad-band and narrow-band interferometer configurations [34,35]. Defining a precise break frequency may not even be necessary. Instead, the GW inspiral templates could be terminated at high frequency in a manner that reproduces the energy spectrum given by a simple analytic form, such as our Eq. (1). The free parameters k_1 and k_2 could then be measured experimentally and compared directly to the predictions of various realistic nuclear EOS used in computing binary equilibrium sequences. In future work, we plan to compute such sequences, and the corresponding energy curves, for a wide variety of published, realistic NS EOS.

We are grateful to Kip Thorne for originally pointing out the importance of the break frequencies in GW spectra. This work was supported by NSF Grants No. PHY-0070918, No. PHY-0121420, and No. PHY-0133425.

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