Fluctuations of Fluctuation-Induced Casimir-Like Forces

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The force experienced by objects embedded in a correlated medium undergoing thermal fluctuations — the so-called fluctuation-induced force — is actually itself a fluctuating quantity. Using a scalar field model, we compute the corresponding probability distribution and show that it is a Gaussian centered on the well-known Casimir force, with a nonuniversal standard deviation that can be typically as large as the mean force itself. The relevance of these results to the experimental measurement of fluctuation-induced forces in soft condensed matter is discussed, as well as the influence of the finite temporal resolution of the measuring apparatus.

In 1948, Casimir predicted that two uncharged conducting plates facing each other in vacuum are subject to a long-range universal attraction [1]. This attraction is due to the modification of the quantum electromagnetic fluctuation spectrum due to the boundary conditions imposed by the conducting plates. In the past 15 years, the concept of Casimir force has been extended to thermally excited elastic forces between objects embedded in a medium with scale-free fluctuations altered by the presence of the objects [2]. Soft matter examples comprise inclusions in complex fluids undergoing thermal fluctuations [2], e.g., fluid membranes [3,4] and critical mixtures [5], or interfaces bounding complex fluids, e.g., liquid crystals [6,7] and superfluids [8,9]. One of the main features of these Casimir interactions is their universality. In a given geometry and at a given temperature, the fluctuation-induced forces depend only on the universality class of the fluctuating medium and on the nature of the imposed boundary conditions (the material's elastic constants and the coupling strength of the boundaries are irrelevant).

While the development of micro- and nanoscale experiments has allowed precise direct verifications of the original vacuum's Casimir force [10–12], experimental characterization of thermal fluctuation-induced forces in soft matter has been scarce [13,14]. The main difficulty with measuring these forces is usually imputed to the presence of stronger background elastic and van der Waals forces. Another important point which has been overlooked is that the fluctuation-induced force is itself a fluctuating quantity: what is commonly referred to as the "Casimir force" is only its *ensemble average*. The fluctuating nature of this force should be taken into proper account before its complete characterization can be achieved. Similar consideration holds for the experimental measurement of quantum Casimir forces [11,12,15] on the classical regime.

In this Letter, we attempt to characterize the fluctuations of thermally excited elastic forces in condensed matter. Generically, we consider in a space of dimension d a fluctuating medium described by a scalar field ϕ with an elastic energy density proportional to $(\vec{\nabla}\phi)^2$. For instance, ϕ could represent the deviation of the director of a nematic liquid crystal [16] from a uniform orientation (in the one-elastic-constant approximation) [6] or the composition of a mixture at its critical point [9].

We consider two plates of lateral extension L separated by a distance H (see Fig. 1) that impose $\phi = 0$ on their boundary. Our main result [Eq. (15) below] is that the force on each plate is a Gaussian variable (for $d > 1$), of average the universal Casimir force $\langle F \rangle \sim k_\mathrm{B} T L^{d-1}/H^d$, and of nonuniversal variance $(\Delta F)^2$. Its dispersion can be characterized by the "noise-over-signal ratio" $\Delta F/\langle F \rangle$. By calculating ΔF we show that this ratio scales as:

$$\frac{\Delta F}{\langle F \rangle} \sim \left(\frac{H}{L}\right)^{(d-1)/2}\left(\frac{H}{a}\right)^{(d+1)/2}. \tag{1}$$

In the above result, a corresponds to a microscopic length scale at which either the continuum description breaks down or the boundary conditions are not efficiently set by the plates [17]. We also consider the dynamics of the field and the role of the finite temporal resolution τ of a measurement apparatus. If τ_a is the microscopic time
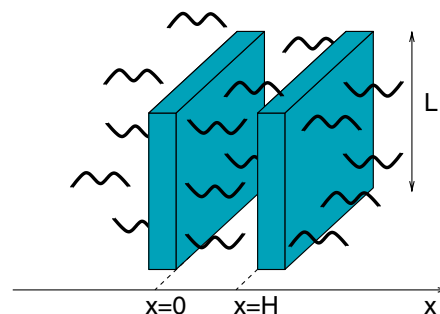

FIG. 1 (color online). Two identical macroscopic plates normal to the x axis immersed in a fluctuating correlated medium.

scale associated with the microscopic length a, we show that the noise to signal ratio is significantly reduced if the apparatus' resolution induces some additional averaging (i.e., $\tau > \tau_a$). More precisely, the average is still $\langle F \rangle$, but the noise to signal ratio now scales as:

$$\frac{\Delta F_\tau}{\langle F_\tau \rangle} \sim \left(\frac{H}{L}\right)^{(d-1)/2} \left(\frac{H}{a}\right)^{(d+1)/2} \sqrt{\frac{\tau_a}{\tau}}. \tag{2}$$

The relevance of the above results to experimental situations is discussed at the end of this Letter.

We consider an elastic Hamiltonian of the form

$$\mathcal{H}[\phi] = \frac{K}{2} \int d^d \vec{R}[\vec{\nabla}\phi(\vec{R})]^2, \tag{3}$$

with $\vec{R} = (x, \mathbf{r})$, and x the coordinate in the direction normal to the plates. We assume that the plates impose Dirichlet boundary conditions, i.e., $\phi(x = 0, \mathbf{r}) = \phi(x = H, \mathbf{r}) = 0$. The whole system is in contact with a heat bath imposing the temperature T. For this minimal model, the free energy $\mathcal{F}(H)$ of the system can be exactly calculated as a function of the interplate distance H [8]. The Casimir force $\langle F(H) \rangle \equiv -\partial\mathcal{F}/\partial H$, i.e., the ensemble average of the fluctuation-induced force, is given by the universal formula

$$\langle F \rangle = -k_B T A_d \frac{L^{d-1}}{H^d}, \tag{4}$$

where the prefactor $A_d = (d-1)\Gamma(d/2)\zeta(d)/(4\pi)^{d/2}$ depends only on the spatial dimension d.

We pursue another route to calculate the Casimir interaction, which yields a clear intuitive picture of the origin of the fluctuation-induced force and permits evaluation of its fluctuations. We use the stress tensor associated with the field [18]: $T_{ij} = \mathcal{E}\delta_{ij} - (\partial_i\phi)[\partial\mathcal{E}/\partial(\partial_j\phi)]$, where $\mathcal{E} = \frac{1}{2}K(\vec{\nabla}\phi)^2$ is the energy density associated with Eq. (3). We first restrict our attention to the effect of the interplate medium on the plate located at $x = H$. For a given conformation of the field, the projection $F^<(H)$ on the x axis of the elastic force exerted on the inner side of this plate is given by the integral of T_{xx}, which reduces to

$$F^<(H) = \frac{K}{2} \int d^{d-1}\mathbf{r}[\partial_x\phi(H, \mathbf{r})]^2, \tag{5}$$

since ϕ vanishes all along the plate. Note that this quantity is always positive; therefore, the field always pushes on the plate. Fourier transforming the field in the transverse direction: $\phi_\mathbf{q}(x) = \int d^{d-1}\mathbf{r}\,\phi(x, \mathbf{r})\,e^{-i\mathbf{q}\cdot\mathbf{r}}$ yields $\mathcal{H} = \sum_\mathbf{q} E_\mathbf{q}$ and $F^<(H) = \sum_\mathbf{q} f_\mathbf{q}(H)$ with $E_\mathbf{q} = \frac{1}{2}L^{1-d}K \int dx[\phi_\mathbf{q}(x)(-\partial_x^2 + q^2)\phi_{-\mathbf{q}}]$ and $f_\mathbf{q}(H) = \frac{1}{2}L^{1-d}K|\partial_x\phi_\mathbf{q}(H)|^2$. These expressions suggest an interesting picture: each $E_\mathbf{q}$ can be understood as the energy of a three-dimensional string parametrized by $[x, \text{Re}(\phi_\mathbf{q}), \text{Im}(\phi_\mathbf{q})]$; each of these virtual strings has a line tension proportional to the rigidity K of the medium and is confined around the x axis by a harmonic potential of stiffness proportional to q^2. These lines are pinned at

$x = 0$ and $x = H$ as a consequence of the Dirichlet boundary conditions (see Fig. 2), and the force they exert on the plate located at $x = H$ is precisely $f_\mathbf{q}(H)$. The total partition function of the medium between the plates is the product $\mathcal{Z} = \prod_{|\mathbf{q}|<1/a} \mathcal{Z}_\mathbf{q}$ of those of the independent strings

$$\mathcal{Z}_\mathbf{q} = \int_{\phi_\mathbf{q}(0)=0}^{\phi_\mathbf{q}(H)=0} \mathcal{D}\phi_\mathbf{q}(x)\, e^{-E_\mathbf{q}[\phi_\mathbf{q}]/k_B T}, \tag{6}$$

where a^{-1} is the high wave vector cutoff. The number of noninteracting strings describing our system is from simple mode counting:

$$N_a \sim \left(\frac{L}{a}\right)^{d-1}. \tag{7}$$

Determining the distribution of the elementary forces $f_\mathbf{q}(H)$ is straightforward. Since the free energy associated with the string fields $\phi_\mathbf{q}$ is quadratic, the distribution of $\partial_x\phi_\mathbf{q}(H)$ is a Gaussian with zero mean and variance $\partial_x\partial_y G_\mathbf{q}(H, H)$, where $G_\mathbf{q}(x, y) \equiv \langle\phi_\mathbf{q}(x)\phi_{-\mathbf{q}}(y)\rangle$ is the two-point correlation function [19]:

$$G_\mathbf{q}(x, y) = L^{d-1} \frac{k_B T}{K} \frac{\sinh[q(H-u)]\sinh(qv)}{q\sinh(qH)}, \tag{8}$$

where $u = \max(x, y)$ and $v = \min(x, y)$. Since the distribution of $\partial_x\phi_\mathbf{q}$ is Gaussian, the fluctuating elementary forces follow a χ^2 distribution:

$$\mathcal{P}\left[\frac{f_\mathbf{q}(H)}{\langle f_\mathbf{q}(H)\rangle} = f\right] = \frac{\theta(f)}{\sqrt{2\pi f}}\exp\left(-\frac{f}{2}\right), \tag{9}$$

where θ is the Heaviside step function, and

$$\langle f_\mathbf{q}(H)\rangle = \frac{k_B T}{2}\left[\frac{2}{\pi a} - q\coth(qH)\right]. \tag{10}$$

The cutoff a^{-1} appears in Eq. (10) because of the discontinuity of the derivative of $G_\mathbf{q}(x, y)$ [20], as a manifestation of the importance of the short wavelength fluctuations. From Eq. (8) follows that the elementary forces undergo large fluctuations: their standard deviation, given by

$$\Delta f_\mathbf{q}(H) = \sqrt{2}\langle f_\mathbf{q}(H)\rangle, \tag{11}$$

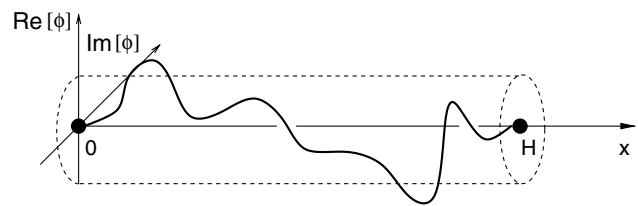compares with their average value.

FIG. 2.   Typical configuration of a fluctuating string pinned at $x = 0$ and $x = H$. The dashed cylinder sketches the effective *cage* imposed by the confining quadratic potential of curvature Kq^2/L^{d-1}.

We can now consider the whole space consisting of three independent subsystems delimited by the plates. The instantaneous force $F(H)$ experienced by the plate at $x = H$ is the difference between two independent stochastic variables $F^<(H) - F^<(\infty)$, where $F^<(\infty)$ obviously describes the contribution of the outer medium $x > H$. Within the string picture, $F(H) = \sum_{\mathbf{q}}[f_{\mathbf{q}}(H) - f_{\mathbf{q}}(\infty)]$ thus appears as the sum of N_a independent variables.

Mean Casimir force.—From Eq. (10), the mean force $\langle F \rangle$ experienced by each plate is given by (replacing the discrete sum by an integral and disregarding the cutoff as the integral converges): $\langle F \rangle = \frac{1}{2} k_B T (L/2\pi)^{d-1} \int d^{d-1}\mathbf{q}[q - q\coth(qH)]$, which yields the result quoted in Eq. (4) above. We thus recover the usual expression of the Casimir force, classically obtained by differentiation of the free energy.

The cancellation of the nonuniversal cutoff contribution in the former sum is easily understood in terms of the string picture: since a string with label $\mathbf{q}$ has a correlation length $\simeq q^{-1}$, the strings with $q \gg H^{-1}$ do not feel the presence of the second plate and their average contributions on both sides of the plate cancel exactly. Conversely, the force imbalance for each string with $q < 1/H$ scales as $-k_B T/H$, and there are $(L/H)^{d-1}$ such strings, the product yielding the scaling form of the Casimir force in Eq. (4).

Distribution of the fluctuation-induced force.—The variance of the force, $(\Delta F)^2 \equiv \langle F^2 \rangle - \langle F \rangle^2$, is the sum of the $2N_a$ elementary contributions $(\Delta f_{\mathbf{q}})^2$ from both sides of the plate:

$$(\Delta F)^2 = (k_B T)^2 L^{d-1} \times \frac{1}{2} \int \frac{d^{d-1}\mathbf{q}}{(2\pi)^{d-1}}$$
$$\times \left\{ \left[\frac{2}{\pi a} - q\coth(qH) \right]^2 + \left(\frac{2}{\pi a} - q \right)^2 \right\}, \quad (12)$$

where the integral must be limited to $|\mathbf{q}| < a^{-1}$ to be defined. When $H \gg a$, the leading behavior is given by the limit $H \to \infty$, i.e., $(\Delta F)^2 \sim (k_B T)^2 L^{d-1} \int_{q<1/a} [(\pi a)^{-1} - q]^2 q^{d-2} dq$, which yields

$$\Delta F \sim \sqrt{N_a} \frac{k_B T}{a}, \quad (13)$$

where again N_a is the number of transverse Fourier modes defined in Eq. (7) above. Contrary to the Casimir (average) force, the variance of the fluctuation-induced force is not universal and is intrinsically related to the physics ruling the interaction between the elastic medium and the immersed plates (through the cutoff). Since the short wavelength fluctuations of the q strings are mainly responsible for the force fluctuations, the scaling behavior of Eq. (13) is expected to hold for any macroscopic external object with smooth shape at the scale a. The force fluctuations also include a subdominant universal part $\Delta'F$, which can be exactly calculated as

$$\Delta'F = (dA_d)^{1/2}\left(\frac{L}{H}\right)^{(d-1)/2}\frac{k_B T}{H}. \quad (14)$$

This cutoff independent part of the dispersion originates from the $N_H = (L/H)^{d-1}$ strings with $q \lesssim H^{-1}$, as in the case of the mean Casimir force.

The $\sqrt{N_a}$ factor in Eq. (13) can easily be understood from the central limit theorem applied to the extensive system of $2N_a$ noninteracting strings (for $d > 1$), generating a force with mean and standard deviation of order $k_B T/a$. The central limit theorem holds since for all the strings $\Delta f_{\mathbf{q}}(H)/\Delta F \to 0$ as $N_a \to \infty$. As a consequence, the distribution $\mathcal{P}_F$ of the force experienced by the plate at $x = H$ is a Gaussian:

$$\mathcal{P}_F(F = f) = \frac{1}{\sqrt{2\pi}\Delta F}\exp\left[-\frac{(f - \langle F \rangle)^2}{2(\Delta F)^2} \right], \quad (15)$$

with its mean and variance being given by Eqs. (4) and (12), respectively.

To summarize, the net force on a plate is the difference between two uncorrelated Gaussian stochastic processes, each of nonuniversal typical amplitude (mean and deviation) dominated by small wavelength contributions. The average of the net force being the difference between the averages, the small wavelength contributions from each side cancel, leaving only the weaker universal Casimir term. In contrast, variances add up so that the standard deviation of the net force depends on microscopics.

Estimation of the Casimir force dispersion.—Assuming that the instantaneous fluctuation-induced force can be measured with a perfect precision, the noise over signal ratio is thus given by Eq. (1), obtained from Eqs. (4) and (13) using $N_a = (L/a)^{d-1}$. Let us estimate it in a typical experimental situation, e.g., a nematic liquid crystal with a director strongly anchored normal to the plates. In this case, we have $d = 3$ and $a \simeq 1$ nm. If the typical size of the plates is $L \simeq 10$ μm and their separation is $H \simeq 100$ nm, then the amplitude of the fluctuations is a *hundred times* larger than the mean force $\langle F \rangle \simeq 1$ pN. In order to lower the noise over signal ratio to 1, the size of the plates must reach 1 mm.

In all the previous discussions, we have implicitly assumed $d > 1$. It is interesting to notice the singularity of the one-dimensional (1D) case. The net force experienced by the plate can no longer be interpreted as the result of a large number of independent contributions. A 1D elastic medium is indeed equivalent to the sole $q = 0$ string. Consequently, the distribution of forces $F(H)$ and $F(\infty)$ are directly given by Eq. (9) taking the limit $q \to 0$. However, even if the distribution is no longer Gaussian, the scaling of the noise over signal ratio in Eq. (1) still holds in the limit $d \to 1$.

Measurement of fluctuation-induced forces and temporal resolution.—It has just been shown that the equilibrium distribution of fluctuation-induced forces may be extremely broad. Any experimental measurement will, however, provide a filtered signal which is averaged

over the temporal resolution τ of the apparatus. In order to estimate to what extent the fluctuating nature of fluctuation-induced forces may be experimentally revealed, a dynamical description is required. A precise description of the motion of the plates is beyond the scope of this tentative approach. In a simple picture, the measurement apparatus provides a signal $F_\tau(t) = \int_{-\infty}^t \chi(t - t')F(t')dt'$. Its response function χ has a typical decay time τ and is causal and normalized [$\int_0^\infty dt' \chi(t') = 1$]. The limit $\tau \to 0$ would correspond to a perfect apparatus which provides a signal completely described by Eq. (15).

The short wavelength excitations of the strings dominate the fluctuations of the net force as it has been previously shown. When measuring the force, the apparatus averages over $N_\tau \simeq \tau/\tau_a$ independent processes, in which τ_a is the microscopic correlation time of the fluctuations associated to the modes with wave vector $\simeq a^{-1}$ in the x direction. Consequently, we expect the force dispersion ΔF to be lowered by a factor of $1/\sqrt{N_\tau}$, which leads to the result reported above in Eq. (2).

In order to check this analysis, we have studied the simplest case of a local and dissipative dynamics for the $\phi_{\mathbf{q}}$ fields described by the Langevin equations

$$\gamma \partial_t \phi_{\mathbf{q}}(x, t) = K[\partial_x^2 - q^2]\phi_{\mathbf{q}}(x, t) + \xi_{\mathbf{q}}(x, t), \qquad (16)$$

subject to $\phi_{\mathbf{q}}(0, t) = \phi_{\mathbf{q}}(H, t) = 0$. In the above equaion, $\xi_{\mathbf{q}}$ is a Gaussian white noise with zero mean and correlations chosen so as to ensure thermal equilibrium, i.e., $\langle \xi_{\mathbf{q}}(x, t)\xi_{\mathbf{q}'}(x', t') \rangle = 2\gamma k_B T (2\pi)^{d-1} \delta^{d-1}(\mathbf{q} + \mathbf{q}')\delta(x - x')\delta(t - t')$. This set of N_a equations corresponds to the Rouse dynamical equations for the N_a pinned elastic stings undergoing thermal fluctuations [21]. The linearity of the Langevin equations above [Eq. (16)] ensures that the $\phi_{\mathbf{q}}$'s are Gaussian fields, so that it is straightforward to determine the statistical properties of the measured force.

The finite temporal resolution τ does not modify the mean force which remains given by Eq. (4), $\langle F_\tau \rangle = \langle F \rangle$. The dispersion of the measured forces can be expressed as the product of the ideal expressions calculated previously by an attenuation factor:

$$\Delta F_\tau = \Delta F \Upsilon(\tau/\tau_a), \qquad (17)$$

where for this dynamics τ_a is given by $\tau_a = \gamma a^2 / K$. The asymptotic scaling behavior of $\Upsilon(\tau/\tau_a)$ does not depend on the specific form of the response function χ: we have $\Upsilon(0) = 1$ and $\Upsilon(s) \sim 1/\sqrt{s}$ in the limit $s \gg 1$. This simple model of a diffusive dynamics thus clearly corroborates the qualitative analysis that lead to Eq. (2) above.

Let us determine the effect of the finite temporal resolution for the experimental example quoted above ($L = 10 \ \mu$m, $H = 100$ nm, $a = 1$ nm). The order of magnitude of the relaxation time τ_a in an experiment involving a nematic liquid crystal is around $0.1 \ \mu$s at $T = 300$ K. Using a measurement device characterized by $\tau \sim 1$ ms such as an optical tweezer [22], the noise over signal ratio is lowered from 100 to 1.

Generally, Eq. (2) quantifies to what extent a slow measurement device is best suited for experimental observation of the universal part of fluctuation-induced forces. From the above analysis, it also follows that replacing the fluctuation-induced force between small objects by the simple Casimir average when studying their interactions and collective behavior may not be justified if these objects are fast movers.

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