Limit Cycles in Quantum Theories

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Renormalization group limit cycles and more chaotic behavior may be commonplace for quantum Hamiltonians requiring renormalization, in contrast to experience based on classical models with critical behavior, where fixed points are far more common. We discuss the simplest quantum model Hamiltonian identified so far that exhibits a renormalization group with both limit cycle and chaotic behavior. The model is a discrete Hermitian matrix with two coupling constants, both governed by a nonperturbative renormalization group equation that involves changes in only one of these couplings and is soluble analytically.

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In 1971, one of us suggested that renormalization group equations could have limit cycle solutions or more chaotic behavior as well as fixed points when the renormalization group equations involve two or more coupling constants [1]. The 1971 paper did not mention the possibility that renormalization group equations might have limit cycle solutions even for differential equations for only one coupling constant. But in 1993, the two of us defined a simple Hamiltonian that requires the renormalization of just a single bare coupling constant g_Λ, where Λ is the cutoff, and the coupling constant g_Λ was found to decrease steadily as Λ increased, until g_Λ reached $-\infty$, after which it jumped to $+\infty$ and started a new period of steady decrease [2]. However, we did not recognize or comment that this behavior constituted a genuine limit cycle.

In this Letter, we analyze a discretized version of our 1993 model. We demonstrate that it exhibits even more startling behavior because it has a renormalization group that exhibits limit cycles or chaotic behavior, depending on the values chosen for the discrete model's free parameters. The renormalization group equation for the discretized model is soluble analytically, just as in our 1993 continuum model, but this equation itself has a discretized form rather than being a differential equation. In the presence of an ultraviolet cutoff Λ, the characteristic feature of the limit cycle behavior for the model is an infinite set of bound states with point of accumulation at zero energy. Moreover, the bound-state energy eigenvalues approach zero energy as a geometrically decreasing series in absolute magnitude, but this geometric behavior has cutoff-dependent corrections for the lowest negative eigenvalues. In contrast, when the ultraviolet cutoff is taken to infinity, the model is renormalizable and the limit cycle leads to an exact geometric series of bound states with energies extending from negative and arbitrarily large all the way to arbitrarily close to zero. We have not examined the case of chaotic behavior in comparable detail, but towards the end of this Letter we identify the conditions that guarantee chaotic solutions for the model's renormalization group equation.

The model Hamiltonian we published in 1993 has no known direct physical application. But in 1999, Bedaque, Hammer, and van Kolck showed that a three-body Hamiltonian with two- and three-body delta function potentials and a cutoff in momentum space is renormalizable and that the three-body coupling approaches a limit cycle as the cutoff Λ approaches ∞ [3], much as it does in our 1993 model. The model of Bedaque, Hammer, and van Kolck, is applicable to the nuclear energy levels of the triton, although the parameters suitable for representing the triton yield only one bound state. It requires a somewhat different choice of parameters to provide the best illustration of a limit cycle. In this case, the deuteron would have a binding energy of 0 and the triton would have an infinite set of bound states with energies converging toward zero [3]. Bedaque *et al.* built on the earlier work of Thomas [4] and Efimov [5], which was recently reviewed by Nielsen, Fedorov, Jensen, and Garrido [6]. Efimov already recognized the existence of an infinite sequence of three-body bound states accumulating at zero energy when the deuteron has a binding energy of 0. The same models are applicable to the analysis of interactions of three helium atoms, where computations indicate that there are two bound levels, and possibly to interactions of some other three-atom systems as well [7].

In the case of the three-body problem, however, the emergence of the limit cycle can be difficult to follow because of the complexity of the mathematical approximations needed to extract the eigenvalues, while the simplicity of our model enables us to analyze with care how the limit cycle behavior emerges from the eigenvalue spectrum. We should also mention that Bernard and LeClair recently observed a potentially cyclic effect in the flow of couplings in complex two-dimensional models with anisotropic current interactions with two couplings [8].

All these examples involve quantum Hamiltonians subject to renormalization. In principle, limit cycles could have arisen in statistical mechanical applications of the renormalization group as well. But to our knowledge, only one example of a limit cycle has been found despite the vast number of known applications. The example, a rough approximation for a three state Potts model [9] in two dimensions, was proposed and analyzed by Huse [10].

With a cutoff, our discretized model takes the form of a finite size matrix with discrete eigenvalues. The discrete matrix has a diagonal submatrix, plus two off-diagonal pieces with two coupling constants. The Hamiltonian requires renormalization in the limit of infinite cutoff. Just as in the continuum case, the renormalization can be constructed analytically and leads to the limit cycle or chaotic behavior in one of the two couplings, while the second coupling constant stays fixed. However, the analytically obtained limit cycle is defined only for a discrete sequence of cutoffs, rather than for a continuously varying cutoff. Using an alternative formulation of the renormalization group (see below), it is possible to define and compute a renormalization process with a continuously varying cutoff, but our studies with the continuous cutoff variation are not discussed here.

The finite matrix Hamiltonians can be diagonalized numerically, when the cutoff is small enough. The renormalizability of the Hamiltonians we discuss here can be demonstrated to high numerical accuracy with cutoffs small enough to allow numerical diagonalization. We provide the demonstration with a comparison of the eigenvalues of two matrices with two different cutoffs, one of size 37×37, the other of size 42×42.

The Hamiltonian to be used in this Letter has the form

$$H_{mn}(g_N, h_N) = (E_m E_n)^{1/2}[\delta_{mn} - g_N - ih_N s_{mn}], \quad (1)$$

where m and n are integers. For $m = n$, $\delta_{mn} = 1$ and $s_{mn} = 0$. For $m \neq n$, $\delta_{mn} = 0$ and $s_{mn} = (m - n)/|m - n|$. The numbers $E_n = b^n$ with $b > 1$ are eigenvalues of the operator H_0 that has matrix elements $\langle m|H_0|n\rangle = H_{mn}(0, 0)$. The eigenvalues are called kinetic energies of the corresponding eigenstates, $|n\rangle$. The remaining part of the Hamiltonian, $H_I = H - H_0$, is called an interaction, and $H_I(0, 0) = 0$. The largest energy allowed in the dynamics is $\Lambda_N = b^N$, which defines the ultraviolet cutoff so that the subscripts $m, n \leq N$.

The continuous version of this model [2] is recovered in the limit $b \to 1$. The discrete model itself has been discussed in the case $h_N = 0$ using similarity renormalization group idea [11] and Wegner's equation [12,13]. The model requires renormalization and exhibits asymptotic freedom if $h_N = 0$. Hamiltonians of Eq. (1) with $h_N = 0$ can be derived in a number of ways, ranging from a discretization of a nonrelativistic Schrödinger equation for a particle on a plane with a two-dimensional δ potential to a discrete version of the transverse dynamics of partons in quantum field theory.

At first, the model with $h_N \neq 0$ does not appear much different from the one with $h_N = 0$. All Hamiltonians defined by Eq. (1) are Hermitian and have a general ultraviolet logarithmically divergent structure. As we demonstrate below, when h_N is not zero, the model exhibits limit cycle behavior as N goes to infinity (or chaos). We prove this by deriving a renormalization group equation that determines g_{N-1} (used with a cutoff Λ_{N-1}), given g_N and h_N used with cutoff Λ_N, such that the low energy eigenvalues stay fixed.

To introduce the renormalization group analysis, the eigenvalue problem

$$\sum_{n=-\infty}^{N} H_{mn}\psi_n = E\psi_m \quad (2)$$

can be solved for ψ_m, $m \leq N$, assuming that one knows E and using the Gaussian elimination. In the first step, one solves for ψ_N in terms of all other components ψ_n with $n < N$. In the next step, one expresses ψ_{N-1} in terms of components ψ_n with $n < N - 1$, and so on. We carry out the first p such steps assuming E is small enough to be neglected. This will be sufficient to enable us to recognize the existence of limit cycles of period p or less, when they occur.

It is convenient to write the eigenstate components in the form $\psi_n = b^{-n/2}\phi_n$ for all $n \leq N$, and define $\sigma_N = \sum_{n=-\infty}^{N} \phi_n$. The first step of the Gaussian elimination gives then

$$\phi_N = (g_N + ih_N)\sigma_{N-1}/(1 - g_N). \quad (3)$$

Substituting this result into the remaining $N - 1$ equations, one obtains a new set that does not explicitly involve the component with the largest kinetic energy but has a different coupling constant instead. Namely,

$$g_{N-1} = g_N + (g_N^2 + h_N^2)/(1 - g_N), \quad (4)$$

while $h_{N-1} = h_N$. Moreover, Eq. (4) can be simplified. Define angles α_N and β to be

$$\alpha_N = \arctan(g_N/h_N), \quad (5)$$

$$\beta = \beta_N = \arctan(h_N). \quad (6)$$

The simplified equation is

$$\alpha_{N-1} = \alpha_N + \beta. \quad (7)$$

It is now seen that the result of p steps of elimination is

$$g_{N-p} = h_N \tan[\alpha_N + p\beta]. \quad (8)$$

We can now demonstrate the existence of renormalization group limit cycles for the discrete Hamiltonians of Eq. (1). All that is necessary is to choose β to be

π/p; then there is a limit cycle of period p in which g_{N-p} and h_{N-p} are identical to g_N and h_N, respectively, since tan is a periodic function of its argument. For example, with $N = 16$ and $p = 5$, so that $h_{16} = \tan(\pi/5)$, and when one arbitrarily sets $g_{16} = 0.0606$, one obtains a cycle with $g_{15} = 0.626$, $g_{14} = 3.090$, $g_{13} = -1.731$, $g_{12} = -0.441$, $g_{11} = g_{16}$, etc. The numbers illustrate the range of couplings that appear, far outside the perturbative range of $|g| \ll 1$. There is no upper bound on the size of g that one can obtain in the cycle by choosing different values of g_{16}. But how does this limit cycle impact the properties of the Hamiltonian in Eq. (2)?

To truly understand the impacts of the limit cycle on the Hamiltonian, we need to study its eigenstates. For this purpose, we consider a finite version of the Hamiltonian, obtained by imposing a lower cutoff of -25 on the indices m and n. The Hamiltonian now has 42 eigenstates. These eigenstates are easily determined numerically for $b = 2$. The results are shown in the column labeled $\Lambda = 2^{16}$ in Table I. But, according to our elimination process, combined with the limit cycle, the same Hamiltonian with the same coupling constants, the same lower limit on m and n of -25, but with an upper cutoff at $N = 11$, should have the same low energy eigenvalues. To confirm this, the eigenstates of the Hamiltonian with an upper cutoff of $N = 11$ are shown in the column labeled $\Lambda = 2^{11}$. Table I demonstrates that the eigenvalues are indeed identical—to six significant figures—as long as they lie below 0.02 in magnitude, and identical to three significant figures if they lie below about 100 in magnitude.

We have one more comparison to make between the two Hamiltonians. If we were to change the lower cutoff from -25 to -30 on the Hamiltonian with upper cutoff $N = 11$, then the second Hamiltonian would differ from the first only in a scale shift. That is, if every element of the first Hamiltonian is multiplied by $1/32$, the result is equal to the second Hamiltonian matrix with a lower cutoff at -30. This means that if the eigenvalues of the first Hamiltonian are multiplied by $1/32$, the result is the eigenvalues of the second Hamiltonian with the changed lower cutoff. But the large eigenvalues of the second Hamiltonian are unaffected by where the lower cutoff sits and, hence, must be essentially the same as $1/32$ times the eigenvalues of the first Hamiltonian even when the lower cutoff of the second Hamiltonian is -25. This is also easily verified from Table I. In particular, all eigenvalues of the second Hamiltonian in Table I that are larger in magnitude than 0.001 are identical to $1/32$ times an eigenvalue of the first Hamiltonian to at least five significant figures.

Given the two comparisons between the two Hamiltonians of Table I, we can infer a single comparison between the eigenstates of the first Hamiltonian in isolation. Namely, for eigenvalues well below the upper cutoff ($\Lambda_{16} = 65\,536$) and well above the lower cutoff (about 0.000 000 03), they should come in groups of five

TABLE I. The columns contain: n—the number of the eigenvalue in the ascending order (positive eigenvalues with $n = 9, 10, \ldots, 18$ are not explicitly discussed in the text and are omitted here for brevity); $\Lambda = 2^{16}$—the eigenvalues of the Hamiltonian with $N = 16$, $M = -25$, and $b = 2$ (see the text for details); $\Lambda = 2^{11}$—the eigenvalues of H after five discrete renormalization group steps (with the same M).

n	$\Lambda = 2^{16}$	$\Lambda = 2^{11}$
42	$0.954734 \times 10^{+5}$	
41	$0.328953 \times 10^{+5}$	
40	$0.131845 \times 10^{+5}$	
39	$0.545303 \times 10^{+4}$	
38	$0.228087 \times 10^{+4}$	$0.298354 \times 10^{+4}$
37	$0.956198 \times 10^{+3}$	$0.102798 \times 10^{+4}$
36	$0.401063 \times 10^{+3}$	$0.412015 \times 10^{+3}$
35	$0.168593 \times 10^{+3}$	$0.170407 \times 10^{+3}$
34	$0.709615 \times 10^{+2}$	$0.712770 \times 10^{+2}$
33	$0.298253 \times 10^{+2}$	$0.298812 \times 10^{+2}$
32	$0.125234 \times 10^{+2}$	$0.125332 \times 10^{+2}$
31	$0.526682 \times 10^{+1}$	$0.526853 \times 10^{+1}$
30	$0.221724 \times 10^{+1}$	$0.221755 \times 10^{+1}$
29	$0.931986 \times 10^{+0}$	$0.932040 \times 10^{+0}$
28	$0.391347 \times 10^{+0}$	$0.391357 \times 10^{+0}$
27	$0.164586 \times 10^{+0}$	$0.164588 \times 10^{+0}$
26	0.692886×10^{-1}	0.692889×10^{-1}
25	0.291245×10^{-1}	0.291245×10^{-1}
24	0.122296×10^{-1}	0.122296×10^{-1}
23	0.514332×10^{-2}	0.514332×10^{-2}
22	0.216526×10^{-2}	0.216526×10^{-2}
21	0.910135×10^{-3}	0.910135×10^{-3}
20	0.382169×10^{-3}	0.382169×10^{-3}
19	0.160723×10^{-3}	0.160723×10^{-3}
8	-0.622117×10^{-6}	-0.622117×10^{-6}
7	-0.206506×10^{-4}	-0.206506×10^{-4}
6	-0.661561×10^{-3}	-0.661561×10^{-3}
5	-0.211707×10^{-1}	-0.211708×10^{-1}
4	$-0.677466 \times 10^{+0}$	$-0.677580 \times 10^{+0}$
3	$-0.216826 \times 10^{+2}$	$-0.217999 \times 10^{+2}$
2	$-0.697598 \times 10^{+3}$	$-0.846472 \times 10^{+3}$
1	$-0.270871 \times 10^{+5}$	

that differ, one from the next, by a factor very close to $1/32$. This is confirmed by Table I. Each group of five eigenstates contains a subgroup of four positive eigenvalues, which differ internally by a ratio of roughly $r = 2^{5/4}$, and one negative eigenvalue. For example, see the group of eigenvalues with numbers 20, 21, 22, 23, and 6, and another group with 24, 25, 26, 27, and 5. The ratios of adjacent positive eigenvalues in the first group are $E_{20}/E_{21} = 0.998\,705/r$, $E_{21}/E_{22} = 0.999\,730/r$, and $E_{22}/E_{23} = 1.001\,280/r$; and in the second group are $E_{24}/E_{25} = 0.998\,714/r$, $E_{25}/E_{26} = 0.999\,733/r$, $E_{26}/E_{27} = 1.001\,280/r$. At the same time, the ratios at the boundaries of the groups are $E_{19}/E_{20} = 1.000\,250/r$, $E_{23}/E_{24} = 1.000\,270/r$, and $E_{27}/E_{28} = 1.000\,270/r$. Using the numbers given in Table I, one obtains

$E_{23}/E_{27} = 1.000\,001/32$, $E_{22}/E_{26} = 0.999\,996/32$, $E_{21}/E_{25} = 0.999\,994/32$, and $E_{20}/E_{24} = 0.999\,984/32$. It is thus evident from Table I that the intermediate positive eigenvalues form groups of four, with all ratios within 1% of r, that repeat to an accuracy reaching six significant figures. One can also evaluate $E_6/E_{20} = -1.731\,07$ and $E_5/E_{24} = -1.731\,10$, while $E_6/E_5 = 0.999\,965/32$, and see that the negative eigenvalues belong to the recurring groups of five.

That the ratios of consecutive positive eigenvalues, for intermediate energies, are close to $r = 2^{5/4} = b^{p/(p-1)}$ is a surprise. The only requirement we can derive for intermediate energies is that the product of 4 (i.e., $p - 1$) consecutive ratios should be 2^5 (i.e., b^p), apart from very small numerical errors. This is verified, to five significant figures, for energy eigenvalues between 0.002 and 0.16 in the case of the first Hamiltonian. But the appearance of r in this pattern seems to be linked to the size of b. When b is made smaller than 2, the ratios become even closer to r. When b is made much larger than 2, then one can show that the ratios of consecutive positive eigenvalues become very different from r, because to leading order for large b, the eigenvalues are directly determined by the sequence of couplings g_N:

$$E_N \sim b^N(1 - g_N), \tag{9}$$

for any N.

The bound states of the model Hamiltonian of this Letter have different behaviors in two limits: the continuum limit $b \to 1$ and the large-b limit. In the continuum limit, one recovers the cyclic behavior of the coupling computed already in [2], and it can be shown now that that behavior was associated with the finite bound-state eigenvalues of the renormalized theory forming a geometric series running from 0 all the way to negative infinity. Denote the ratio of adjacent bound-state energies by $\tilde{r}$. For b near 1, we find that the discrete Hamiltonian continues to have just one bound state for each period of the limit cycle, and, hence, the ratio $\tilde{r}$ is equal to b^p. As $b \to 1$, p must go to infinity to keep the ratio of energy eigenvalues fixed. This means that h_N becomes approximately equal to π/p and goes to zero in the continuum limit. But h_N goes to zero in such a way that allows the discrete sum of states of Eq. (2) to be replaced by a continuum integration with nonzero couplings. The details of this continuum limit are straightforward and are omitted.

In the opposite limit of very large b, the number of bound states per cycle becomes more arbitrary. There is a bound state for each coupling g_N within a cycle that is greater than 1. It is easily verified that in the case $p = 5$, either one or two eigenvalues per cycle will be negative, depending on the value of the initial coupling for the largest N. As p becomes very large, typically a quarter of the eigenvalues are negative. But we also note that, for any value of b greater than 1, there can be chaotic solutions to the renormalization group as well as limit cycle solutions. Chaotic solutions occur whenever β/π is an irrational number, in which case the sequence of couplings g_N has no finite period p at all. The possibility of chaotic solutions to renormalization group equations was noted in [1].

So far, we have focused on a single model Hamiltonian with a simple one dimensional renormalization group. The model has a Hermitian Hamiltonian matrix with a nonzero imaginary skew-symmetric part that contributes to logarithmic divergences. In this circumstance, the renormalization group exhibits a limit cycle behavior and leads to subsets of eigenvalues that recur in a geometric series, with each subset of eigenvalues decreasing by a constant factor relative to the prior subset, and each subset contains negative bound-state eigenvalues. The full set of negative eigenvalues ranges from zero to negative infinity in the renormalized theory. Is this model unique and exceptional in its departure from the fixed point behavior, or is the departure a generic phenomenon shared with many more quantum Hamiltonians? The appearance of a limit cycle in three-body problems of quantum mechanics [3,7] provides a far less trivial example of the cycle for study and may be a warning that more such examples could surface in the future. The similarity renormalization group procedure for quantum Hamiltonians [11] can be implemented using the generic and elegant Wegner flow equation [12,13] and provides one starting point for further study of this question. Finally, while we discussed here only the limit cycle case in detail, the possibility of chaotic behavior ensures that further investigation of the model is also of interest concerning fundamental issues beyond limit cycles in quantum theories.

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[1] K. G. Wilson, Phys. Rev. D **3**, 1818 (1971).
[2] St. D. Głazek and K. G. Wilson, Phys. Rev. D **47**, 4657 (1993).
[3] P. F. Bedaque, H.-W. Hammer, and U. van Kolck, Phys. Rev. Lett. **82**, 463 (1999).
[4] L. H. Thomas, Phys. Rev. **47**, 903 (1935).
[5] V. Efimov, Phys. Lett. **33B**, 563 (1970).
[6] E. Nielsen, D.V. Fedorov, A. S. Jensen, and E. Garrido, Phys. Rep. **347**, 373 (2001).
[7] E. Braaten and H.-W. Hammer, cond-mat/0203421.
[8] D. Bernard and A. LeClair, Phys. Lett. B **512**, 78 (2001).
[9] R. B. Potts, Proc. Cambridge Philos. Soc. **48**, 106 (1952).
[10] D. A. Huse, Phys. Rev. B **24**, 5180 (1981).
[11] St. D. Głazek and K. G. Wilson, Phys. Rev. D **48**, 5863 (1993); **49**, 4214 (1994).
[12] F. Wegner, Ann. Phys. (Leipzig) **3**, 77 (1994).
[13] St. D. Głazek and K. G. Wilson, Phys. Rev. D **57**, 3558 (1998).