Fractal Structures of Normal and Anomalous Diffusion in Nonlinear Nonhyperbolic Dynamical Systems

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A paradigmatic nonhyperbolic dynamical system exhibiting deterministic diffusion is the smooth nonlinear climbing sine map. We find that this map generates fractal hierarchies of normal and anomalous diffusive regions as functions of the control parameter. The measure of these self-similar sets is positive, parameter dependent, and in case of normal diffusion it shows a fractal diffusion coefficient. By using a Green-Kubo formula we link these fractal structures to the nonlinear microscopic dynamics in terms of fractal Takagi-like functions.

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One of the most fundamental questions in nonequilibrium statistical mechanics is to understand transport processes starting from first principles, that is, by analyzing the microscopic nonlinear equations of motion of a many-particle system. This is somewhat in contrast to conventional methods of statistical mechanics, which rely on the assumption of stochastic randomness in the particle dynamics. Linking macroscopic transport to microscopic deterministic chaos thus requires one to connect statistical mechanics with dynamical systems theory [1–3]. Much was learned by applying such combined methods to a hierarchy of simple model systems that consists of low-dimensional chaotic maps [4–8], particle billiards [3,9,10], and nonlinear pendulum equations [11–13]. The latter differential equations were in turn successfully used to describe deterministic diffusion in experiments on dissipative systems driven by periodic forces such as Josephson junctions in the presence of microwave radiation [14], superionic conductors [15], and systems exhibiting charge-density waves [16].

In this Letter we focus on the one-dimensional so-called [17] *climbing sine map* [4,5], which is obtained from a driven pendulum equation in the limit of strong dissipation via discretization of time [18]. This map is a typical example of a *nonhyperbolic* dynamical system that exhibits a rich dynamics consisting of chaotic diffusive motion, ballistic dynamics, and localized orbits. Under parameter variation these dynamical regimes are highly intertwined resulting in complex scenarios related to the appearance of periodic windows [5]. It is now interesting to relate the climbing sine map to one-dimensional *hyperbolic* maps sharing the same symmetries. These maps are purely normal diffusive; however, here the diffusion coefficient was found to be a fractal function of control parameters [19,20]. This phenomenon was conjectured to be typical for low-dimensional periodic chaotic dynamical systems exhibiting normal transport [19,20] and was later on detected for other transport coefficients [21], and in more complicated models [9,10].

Up to now the fractality of transport coefficients could be assessed for hyperbolic systems only. Hence, a crucial question is whether the origin of normal and anomalous diffusion in the broad class of nonhyperbolic systems is as well of a fractal nature. In this Letter we show that nonhyperbolic behavior not only amplifies such fractal structures but generates even more complex fractal characteristics of deterministic diffusion under parameter variation.

The climbing sine map we study is defined as

$$X_{n+1} = M_a(X_n), \qquad M_a(X) := X + a\sin(2\pi X), \tag{1}$$

where $a \in \mathbb{R}$ is a control parameter, $X \in \mathbb{R}$, and X_n is the position of a point particle at discrete time n. Obviously, $M_a(X)$ possesses translation and reflection symmetry,

$$M_a(X + p) = M_a(X) + p, \qquad M_a(-X) = -M_a(X). \tag{2}$$

The periodicity of the map naturally splits the phase space into different cells $(p, p+1]$, $p \in \mathbb{Z}$. We focus on parameters $a > 0.732\,644$ for which the extrema of the map exceed the boundaries of each cell for the first time indicating the onset of diffusive motion.

The bifurcation diagram of the associated circle map $m_a(x) := M_a(X)\mathrm{mod}1$, $x := X\mathrm{mod}1$ consists of infinitely many periodic windows; see Fig. 1. Whenever there is a window the dynamics of Eq. (1) is either ballistic or localized [5]. Figure 1 demonstrates that this scenario has a strong impact on the diffusion coefficient defined by $D(a) := \lim_{n\to\infty}\langle X_n^2\rangle/(2n)$, where the brackets denote an ensemble average over moving particles. For localized dynamics orbits are confined within some finite interval in phase space implying subdiffusive behavior for which the diffusion coefficient vanishes, whereas for ballistic motion particles propagate superdiffusively with the diffusion coefficient being proportional to n. Only for normal diffusion $D(a)$ is nonzero and finite. At the boundaries of each periodic window there is transient

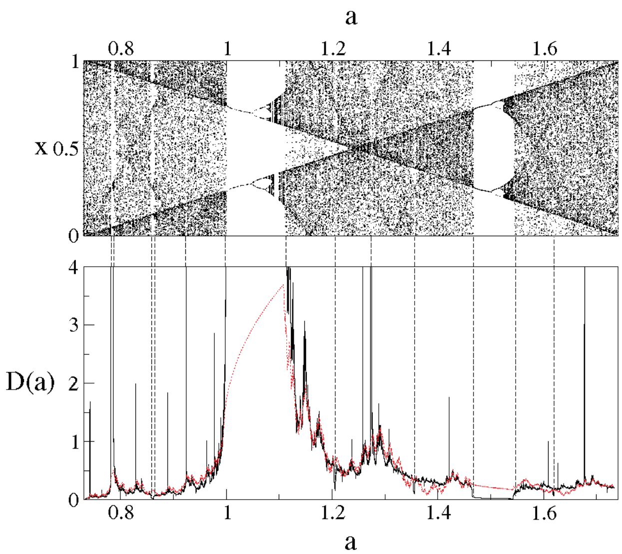

FIG. 1 (color online). Upper panel: bifurcation diagram for the climbing sine map. Lower panel: diffusion coefficient from simulations as a function of the control parameter a in comparison with the correlated random walk approximation $D_{10}^1(a)$ (dots). The dashed vertical lines connect regions of anomalous diffusion, $D(a) \to \infty$ or $D(a) \to 0$, with ballistic and localized dynamics in respective windows of the bifurcation diagram. All quantities here and in the following figures are without units.

intermittentlike behavior eventually resulting in normal diffusion with $D(a) \sim a^{(\pm 1/2)}$ [4,5]. Here we are interested in the complete parameter-dependent diffusion coefficient. For this purpose we compute $D(a)$ from numerical simulations by using the Green-Kubo formula for maps [3,10,19,21],

$$D_n(a) = \langle j_a(x) J_a^n(x) \rangle - \tfrac{1}{2}\langle j_a^2(x) \rangle, \qquad (3)$$

where the angular brackets denote an average over the invariant density of the circle map, $\langle \cdots \rangle := \int dx \rho(x) \ldots$. The jump velocity j_a is defined by $j_a(x_n) := [X_{n+1}] - [X_n] \equiv [M_a(x_n)]$, where the square brackets denote the largest integer less than the argument. The sum $J_a^n(x) := \sum_{k=0}^n j_a(x_k)$ gives the integer value of the displacement of a particle after n time steps that started at some initial position $x \equiv x_0$ called jump velocity function. Equation (3) defines a time-dependent diffusion coefficient which, in case of normal diffusion, converges to $D(a) \equiv \lim_{n\to\infty} D_n(a)$. In our simulations we truncated $J_a^n(x)$ after having obtained enough convergence for $D(a)$, that is, after 20 time steps. The invariant density was obtained by solving the continuity equation for $\rho(x)$ with the histogram method of Ref. [1].

The highly nontrivial behavior of the diffusion coefficient in Fig. 1 can qualitatively be understood as follows: The Green-Kubo formula Eq. (3) splits the dynamics into an intercell dynamics, in terms of integer jumps, and into an intracell dynamics, as represented by the invariant density. We first approximate the invariant density in Eq. (3) to $\rho(x) \simeq 1$ irrespective of the fact that it is a complicated function of x and a [5]. This approximate diffusion coefficient we denote with a superscript in Eq. (3), $D_n^1(a)$. The term for $n = 0$ is well known as the stochastic random walk approximation for maps, which excludes any higher-order correlations [4,5,20]. The generalization $D_n^1(a)$, $n > 0$ was called correlated random walk approximation [10]. We now use this systematic expansion to analyze the diffusion coefficient of the climbing sine map in terms of higher-order correlations.

In Fig. 2(a) we depict results for $D_n(a)$ at $n = 1, \ldots, 10$. One clearly observes convergence of this approximation in parameter regions with normal diffusion. Indeed, a comparison of $D_{10}^1(a)$ with $D(a)$, as shown in Fig. 1, demonstrates that there is qualitative agreement on large scales. On the other hand, for parameters corresponding to ballistic motion the sequence of $D_n^1(a)$ diverges, in agreement with $D(a) \to \infty$, whereas for localized dynamics it alternates between two solutions. This oscillation is reminiscent of the dynamical origin of localization in terms of certain period-two orbits. That these solutions are nonzero is due to the fact that the invariant density was approximated. In regions of normal diffusion this approximation nicely reproduces the irregularities in the diffusion coefficient. Even more importantly, the magnifications in Fig. 2 give clear evidence for a self-similar structure of the diffusion coefficient.

We now further analyze the dynamical origin of these different structures. According to its definition, the time-dependent jump velocity function $J_a^n(x)$ fulfills the recursion relation

$$J_a^n(x) = j_a(x) + J_a^{n-1}(m_a(x)). \qquad (4)$$

$J_a^n(x)$ is getting extremely complicated after some time steps; thus we introduce the more well-behaved function

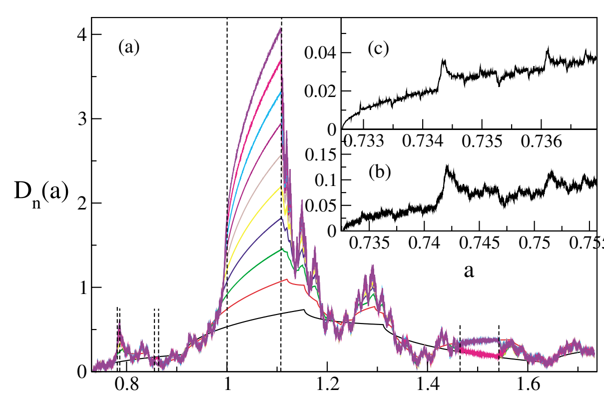

FIG. 2 (color online). (a) Sequence of correlated random walks $D_n^1(a)$ for $n = 1, \ldots, 10$. The dashed lines define the same periodic windows as in Fig. 1. Insets (b) and (c) contain blowups of $D_{10}^1(a)$ in the initial region of (a). They show self-similar behavior on smaller and smaller scales.

$$T_a^n(x) := \int_0^x J_a^n(z)dz, \qquad T_a^n(0) \equiv T_a^n(1) \equiv 0. \tag{5}$$

Integration of Eq. (4) then yields the recursive functional equation

$$T_a^n(x) = t_a(x) + \frac{1}{m_a'(x)} T_a^{n-1}(m_a(x)) - I(x) \tag{6}$$

containing the integral term

$$I(x) := \int_0^{m_a(x)} dz g''(z) T_a^{n-1}(z), \tag{7}$$

where $t_a(x) := \int dz j_a(z)$, $m_a'(x) := dm_a(x)/dx$, and $g''(z)$ is the second derivative of the inverse function of $m_a(x)$ [22]. For piecewise linear hyperbolic maps $I(x)$ simply disappears and the derivative in front of the second term reduces to the local slope of the map thus recovering ordinary de Rham–type equations [3,20,21]. It is not known to us how to directly solve this generalized de Rham equation for the climbing sine map; however, solutions can alternatively be constructed from Eq. (5) on the basis of simulations. Results are shown in Fig. 3. For normal diffusive parameters the limit $T_a(x) = \lim_{n\to\infty} T_a^n(x)$ exists, and the respective curve is fractal over the whole unit interval somewhat resembling (generalized) fractal Takagi functions [3,20,21]. However, in the case of periodic windows $T_a^n(x)$ either diverges due to ballistic flights or it oscillates indicating localization. Interestingly, in these functions the corresponding attracting sets appear in the form of smooth, nonfractal regions on fine scales, whereas the other regions look fractal.

The diffusion coefficient can now be formulated in terms of these fractal functions by integrating Eq. (3). For $a \in (0.732\,644, 1.742\,726]$ we get

$$D(a) = 2[T_a(x_2)\rho(x_2) - T_a(x_1)\rho(x_1)] - D_0^\rho(a), \tag{8}$$

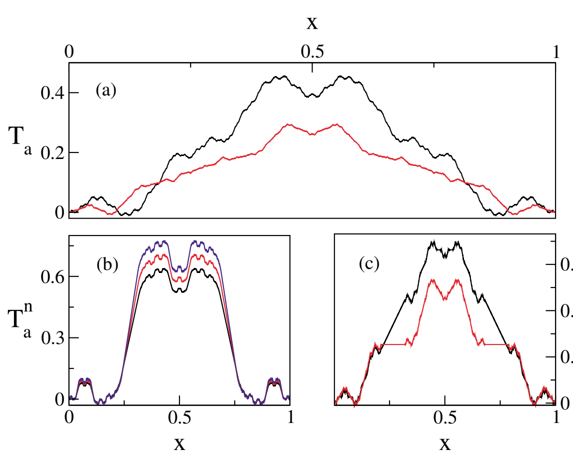

FIG. 3 (color online). Functions $T_a^n(x)$ for the climbing sine map as defined by Eqs. (4)–(7). (a) Diffusive dynamics at $a = 1.2397$ (upper curve) and at $a = 1.7427$ (lower curve), (b) ballistic dynamics at $a = 1.0$, and (c) localized dynamics at $a = 1.5$. In (a) the limiting case $n \to \infty$ is shown; in (b) and (c) it was $n = 5, 6, 7$.

where x_i, $i = 1, 2$ is defined by $[M_a(x_i)] := 1$ and $D_0^\rho(a) := \int_{x_1}^{x_2} dx \rho(x)$. Our previous approximation $D_n^1(a)$ is recovered from this equation as a special case.

The intimate relation between periodic windows and the irregular behavior of the diffusion coefficient motivates us to investigate the structure of the periodic windows in the climbing sine map in more detail. The appearance of windows was analyzed quite extensively for nondiffusive unimodal maps [23], whereas for diffusive maps on the line, apart from the preliminary studies of Ref. [5], nothing appears to be known. The windows are generated by certain periodic orbits; consequently there are infinitely many of them, and they are believed to be dense in the parameter set [2]. Windows with ballistic dynamics are born through tangent bifurcations, further undergo Feigenbaum-type scenarios, and eventually terminate at crisis points. Windows with localized orbits occur only at even periods. They start with tangent bifurcations and exhibit a symmetry breaking at slope-type bifurcation points.

In order to analyze the structure of the regions of anomalous diffusion, we sum up the number of period-six windows as a function of the parameter; that is, the total number is increased by one for any parameter value at which a new period-six window appears. This sum forms a devil's staircase-like structure in parameter space indicating an underlying Cantor set-like distribution for the corresponding anomalous diffusive region; see Fig. 4. The (Lebesque) measure of periodic windows is obviously positive; hence this set must be a fat fractal [24]. Its self-similar structure can quantitatively be assessed by computing the so-called fatness exponent [25]. We are furthermore interested in the parameter dependence of this fractal structure; therefore we divide the parameter line into subsets labeled by the integer value of the map maximum on the unit interval, $[M_a(X_{max})] = j$, $j \in \mathbb{Z}$. For $j = 1, 2, 3$ we obtain a fatness exponent of 0.45 with errors of 0.03, 0.04, and 0.05 for the different j.

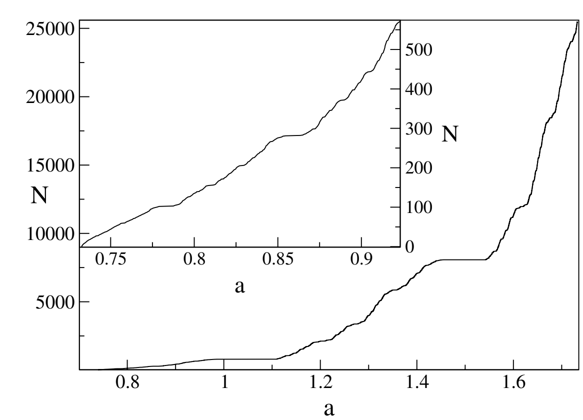

FIG. 4. Devil's staircase-like structure formed by the distribution of periodic windows as a function of the control parameter. N is the integrated number of period-six windows. The inset shows a blowup of the initial region.

We mention that this value was conjectured to be universal and was also obtained for nondiffusive unimodal maps [25].

We now study the measure of the windows as a function of the parameter. For this purpose we computed all windows up to period six for the first subset, up to period five for $j = 2, 3$, and we summed up their measures in the respective subsets. We find that the total measure decays exponentially as a function of j while oscillating with odd and even values of j on a finer scale [26]. This oscillation can be traced back to windows generated by localized dynamics that appear only at even periods thus contributing only periodically to the total measure. However, different measures of "ballistic" and "localized" windows decay with the same rate. We have furthermore computed the complementary measure C_j of diffusive dynamics in the jth subset of parameters. We find that $C_1 = 0.783$, $C_2 = 0.898$, and $C_3 = 0.932$ with an error of ± 0.002, so the measure of the diffusive regions is always nonzero and seems to approach one with increasing parameter values.

We conclude with a few remarks: (i) It would be desirable to perform a spectral analysis of the Frobenius-Perron operator governing the probability density of this map [27]. Combining such an analysis with the Takagi function approach outlined here may lead to a general theory of nonhyperbolic transport. (ii) It might furthermore be interesting to link our work more closely to the stochastic modeling approach of Ref. [6]. (iii) We hope that these results provide some guidelines for studying fractal transport coefficients in more complex models such as the ones in Refs. [8,12,13]. (iv) We finally emphasize the importance to look for possibly fractal transport coefficients in experiments. A promising candidate appears to be the phase dynamics in SQUID's as very recently analyzed theoretically [13] and studied experimentally [28].

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[1] A. Lichtenberg and M. Lieberman, *Regular and Stochastic Motion* (Springer, New York, 1983).
[2] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, England, 1997).
[3] P. Gaspard, *Chaos, Schattering and Statistical Mechanics* (Cambridge University Press, Cambridge, England, 1998); J. R. Dorfman, *An Introduction to Chaos in Non-Equilibrium Statistical Mechanics* (Cambridge University Press, Cambridge, England, 1999).
[4] T. Geisel and J. Nierwetberg, Phys. Rev. Lett. **48**, 7 (1982); S. Grossmann and S. Thomae, Phys. Lett. **97A**, 263 (1983); P. Reimann, Phys. Rev. E **50**, 727 (1994).
[5] M. Schell, S. Fraser, and R. Kapral, Phys. Rev. A **26**, 504 (1982); S. Grossmann and H. Fujisaka, *ibid.* **26**, 1779 (1982); H. Fujisaka and S. Grossmann, Z. Phys. B **48**, 261 (1982).
[6] M. F. Shlesinger, G. M. Zaslavsky, and J. Klafter, Nature (London) **363**, 31 (1993); J. Klafter and G. Zumofen, Phys. Rev. E **49**, 4873 (1994); G. Zumofen and J. Klafter, *ibid.* **51**, 1818 (1995).
[7] R. Artuso, Phys. Lett. A **160**, 528 (1991); I. Dana and M. Amit, Phys. Rev. E **51**, R2731 (1995).
[8] A. B. Rechester and R. B. White, Phys. Rev. Lett. **44**, 1586 (1980); R. Ishizaki *et al.*, Prog. Theor. Phys. **85**, 1013 (1991); G. M. Zaslavsky, M. Edelman, and B. A. Niyazov, Chaos **7**, 159 (1997); P. Leboeuf, Physica (Amsterdam) **116D**, 8 (1998).
[9] R. Klages and C. Dellago, J. Stat. Phys. **101**, 145 (2000); T. Harayama and P. Gaspard, Phys. Rev. E **64**, 036215 (2001).
[10] R. Klages and N. Korabel, J. Phys. A **35**, 4823 (2002).
[11] B.V. Chirikov, Phys. Rep. **52**, 263 (1979); B. A. Huberman, J. P. Crutchfield, and N. H. Packard, Appl. Phys. Lett. **37**, 750 (1980); R. F. Miracky, M. H. Devoret, and J. Clarke, Phys. Rev. A **31**, 2509 (1985); J. A. Blackburn and N. Grønbech-Jensen, Phys. Rev. E **53**, 3068 (1996).
[12] H. Sakaguchi, Phys. Rev. E **65**, 067201 (2002); R. Festa, A. Mazzino, and D. Vincenzi, nlin.CD/0204020.
[13] K. I. Tanimoto, T. Kato, and K. Nakamura, Phys. Rev. B **66**, 012507 (2002).
[14] E. Ben-Jacob *et al.*, Appl. Phys. Lett. **38**, 822 (1981); M. Cirillo and N. F. Pedersen, Phys. Lett. **90A**, 150 (1982); R. F. Miracky, J. Clarke, and R. H. Koch, Phys. Rev. Lett. **50**, 856 (1983); J. M. Martinis and R. L. Kautz, *ibid.* **63**, 1507 (1989).
[15] H. U. Beyeler, Phys. Rev. Lett. **37**, 1557 (1976); S. Martin and W. Martienssen, *ibid.* **56**, 1522 (1986).
[16] S. E. Brown, G. Mozurkewich, and G. Gruner, Phys. Rev. Lett. **52**, 2277 (1984).
[17] If restricted to the circle, this map reduces to the *sine circle map* [2]; however, here we are interested in the spatially continued *lifted* circle map that was coined *climbing sine* in the literature [4,5].
[18] H. Koga, H. Fujisaka, and M. Inoue, Phys. Rev. A **28**, 2370 (1983); T. Bohr, P. Bak, and M. H. Jensen, *ibid.* **30**, 1970 (1984).
[19] R. Klages and J. R. Dorfman, Phys. Rev. Lett. **74**, 387 (1995); Phys. Rev. E 59, 5361 (1999).
[20] R. Klages, *Deterministic Diffusion in One-Dimensional Chaotic Dynamical Systems* (Wissenschaft & Technik Verlag, Berlin, 1996).
[21] P. Gaspard and R. Klages, Chaos **8**, 409 (1998); J. Groeneveld and R. Klages, J. Stat. Phys. **109**, 821 (2002).
[22] In order to define an inverse of $m_a(x)$, we split the unit interval into subintervals on which this function is piecewise invertible.
[23] N. Metropolis, M. L. Stein, and P. R. Stein, J. Comb. Theory **15**, 25 (1973); M.V. Jacobson, Sov. Math. Dokl. **19**, 1452 (1978); M. J. Feigenbaum, J. Stat. Phys. **19**, 25 (1979); **21**, 669 (1979); Physica (Amsterdam) **7D**, 16 (1983).
[24] J. A. Yorke *et al.*, Phys. Rev. Lett. **54**, 1095 (1985).
[25] J. D. Farmer, Phys. Rev. Lett. **55**, 351 (1985).
[26] N. Korabel and R. Klages (unpublished).
[27] D. Alonso *et al.*, Phys. Rev. E **54**, 2474 (1996).
[28] S. Weiss *et al.*, Europhys. Lett. **51**, 499 (2000).