Border between Regular and Chaotic Quantum Dynamics

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We identify a border between regular and chaotic quantum dynamics. The border is characterized by a power-law decrease in the overlap between a state evolved under chaotic dynamics and the same state evolved under a slightly perturbed dynamics. For example, the overlap decay for the quantum kicked top is well fitted with $[1 + (q-1)(t/\tau)^2]^{1/(1-q)}$ (with the nonextensive entropic index q and τ depending on perturbation strength) in the region preceding the emergence of quantum interference effects. This region corresponds to the edge of chaos for the classical map from which the quantum chaotic dynamics is derived.

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Classical chaotic dynamics is characterized by strong sensitivity to initial conditions. Two initially close points move apart exponentially rapidly as the chaotic dynamics evolve. The rate of divergence is quantified by the Lyapunov exponent [1]. At the border between chaotic and nonchaotic regions (the "edge of chaos"), the Lyapunov exponent goes to zero. However, it may be replaced by a generalized Lyapunov coefficient [2] describing power-law, rather than exponential, divergence of classical trajectories.

This Letter identifies a characteristic signature for the edge of quantum chaos. Quantum states maintain a constant overlap fidelity, or distance, under all quantum dynamics, regular and chaotic. One way to characterize quantum chaos is to compare the evolution of an initially chosen state under the chaotic dynamics with the same state evolved under a perturbed dynamics [3–5]. When the initial state is in a regular region of a mixed system, a system with regular and chaotic regions, the overlap remains close to 1. When the initial state is in a chaotic zone, the overlap decay is exponential. This Letter shows that at the edge of quantum chaos there is a region of polynomial overlap decay.

The Lyapunov exponent description of chaos is as follows [1]. If Δx_0 is the distance between two initial conditions, we define $\xi = \lim_{\Delta x_0 \to 0}(\frac{\Delta x_t}{\Delta x_0})$ to describe how far apart two initially arbitrarily close points become at time t. Generally, $\xi(t)$ is the solution to the differential equation $\frac{d\xi(t)}{dt} = \lambda_1 \xi(t)$, such that $\xi(t) = e^{\lambda_1 t}$ (λ_1 is the Lyapunov exponent). When the Lyapunov exponent is positive, the dynamics described by $\xi(t)$ is strongly sensitive to initial conditions and we have chaotic dynamics.

This description of chaos works well for classical Newtonian mechanics, but it cannot hold true for quantum mechanical wave functions governed by the linear Schrödinger equation. Indeed, like the overlap between Liouville probability densities, the overlap between any two quantum wave functions is constant in time. This difficulty has led to the study of "quantum chaos," the search for characteristics of quantum dynamics that manifest themselves as chaos in the classical realm [6–9].

As a possible signature of quantum chaos, Peres [3,4] proposed comparing the evolution of a state under an unperturbed, H, and perturbed, $H + \delta V$, Hamiltonian for chaotic and nonchaotic dynamics. The divergence of the states after a time t is measured via the overlap

$$O(t) = |\langle \psi_u(t) \mid \psi_p(t) \rangle|, \quad (1)$$

where ψ_u is the state evolved under the unperturbed system operator, and ψ_p is the state evolved under the perturbed operator. Recent insights have sharpened the differences between chaotic and regular dynamics under this approach, and several regimes of overlap decay behavior based on perturbation strength have been identified. The overlap decays for a short time as a quadratic. After this time, for chaotic dynamics with weak perturbation the overlap decay is Gaussian, as expected from first order perturbation theory [10–12]. For stronger perturbations, where perturbation theory breaks down, the overlap decay is exponential. This occurs when the magnitude of a typical off diagonal element of V expressed in the ordered eigenbasis of H is greater than the average level spacing of the system, Δ. The regime of exponential overlap decay is called the Fermi golden rule (FGR) regime [12,13]. The rate of the exponential decay will increase with stronger perturbation as the perturbation strength squared until the decay rate reaches a value given by the classical Lyapunov exponent [12,14,15] or the bandwidth of H [12]. The crossover regime from Gaussian to exponential decay has also been studied [13]. We note that many of the works cited use O^2 as the fidelity. Here, we follow [11] and simply use the overlap, O.

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For regular, nonchaotic systems the FGR regime overlap decay is a Gaussian, faster than the exponential decay of chaotic dynamics. This nonintuitive result is explained using a correlation functions formulation of the overlap by Prosen [11]. In addition, a power-law decay $\propto t^{3/2}$ has been found for an integrable system [16].

The initial overlap decay behavior continues until some saturation level [11]. For coherent and random pure states, the saturation level $1/N$ for the exponential decay (in the FGR regime) and $2/N$ for Gaussian decay (in the weak perturbation regime), where N is the dimension of the system Hilbert space. However, for eigenstates of the system and mixed random states, the saturation level increases with increasing perturbation strength [11].

Here we study a mixed system, a system with both chaotic and regular dynamics. Coherent states within the regular regime are practically eigenstates of the system and the overlap of these states oscillates close to unity [4,11]. This is shown in Fig. 1 where the initial coherent state is centered at a fixed point of order one of the regular map. Coherent states in the chaotic regime of the system show exponential overlap decay in the FGR regime and Gaussian overlap decay for weak perturbations. We show that, in both the FGR and weak perturbation regimes, states near the chaotic border have a polynomial overlap decay.

In 1988, one of us [17] introduced in statistical mechanics the generalized entropy form

$$S_q = k\frac{1}{q-1}\left(1-\sum_{i=1}^{W} p_i^q\right), \quad (2)$$

where k is a positive constant, p_i is the probability of finding the system in microscopic state i, and W is the number of possible microscopic states of the system; q is the entropic index which characterizes the degree of the system nonextensivity. In the limit $q \longrightarrow 1$, we recover the usual Boltzmann entropy

$$S_1 = -k\sum_{i=1}^{W} p_i \ln p_i. \quad (3)$$

To demonstrate that q characterizes the degree of the system nonextensivity, it is useful to examine the S_q entropy addition rule [18]. If A and B are two independent systems such that the probability $p(A+B) = p(A)p(B)$, the entropy of the total system $S_q(A+B)$ is given by the following equation:

$$\frac{S_q(A+B)}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1-q)\frac{S_q(A)S_q(B)}{k^2}. \quad (4)$$

From the above equation it is realized that $q < 1$ corresponds to superextensivity and $q > 1$ to subextensivity. Using this entropy to generalize statistical mechanics and thermodynamics has helped explain many natural phenomena in a wide range of fields.

One application of this nonextensive entropy occurs in one-dimensional dynamical maps. As explained above, when the Lyapunov exponent is positive, the system dynamics is strongly sensitive to initial conditions and is characterized as chaotic dynamics. When the Lyapunov exponent is zero, it has been conjectured [2] (and proven [19] for the logistic map) that the distance between two initially arbitrarily close points is described by $\frac{d\xi}{dt} = \lambda_{q_{\rm sen}}\xi^{q_{\rm sen}}$ leading to $\xi = [1 + (1-q_{\rm sen})\lambda_{q_{\rm sen}} t]^{1/(1-q_{\rm sen})}$ ("sen" stands for sensitivity). This requires the introduction of $\lambda_{q_{\rm sen}}$ as a generalized Lyapunov coefficient. The Lyapunov coefficient scales inversely with time as a power law instead of the characteristic exponential of a Lyapunov exponent. Thus, there exists a regime, $q_{\rm sen} < 1$, $\lambda_1 = 0$, $\lambda_{q_{\rm sen}} > 0$, which is weakly sensitive to initial conditions and is characterized by having power-law, instead of exponential, mixing. This regime is called the edge of chaos.

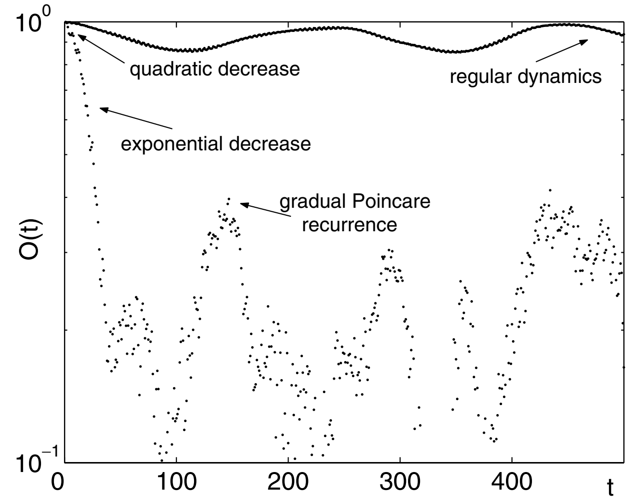

FIG. 1. Overlap versus time for initial angular momentum coherent states located in the chaotic region and the regular region of the quantum kicked top. The system has a spin 120 and is evolved under the kicked top Hamiltonian with $\alpha = 3$ and $\alpha' = 3.015$. The overlap of the state in the chaotic region decreases exponentially with the number of iterations of the map. The state in the regular region is practically an eigenstate of the system and therefore oscillates close to unity.

The polynomial overlap decay found for initial states of a mixed system near the chaotic border are at the "edge of quantum chaos," the border between regular and chaotic quantum dynamics. This region is the quantum parallel of the region characterized classically by the generalized Lyapunov coefficient.

The system studied is the quantum kicked top (QKT) [20] defined by the unitary operator:

$$U_{\rm QKT} = e^{-i\pi J_y/2\hbar} e^{-i\alpha J_z^2/2j\hbar}. \quad (5)$$

j is the angular momentum of the top and α is the "kick" strength. We use a QKT with $\alpha = 3$ whose classical analog has a mixed phase space, regions of chaotic and regular dynamics. The perturbed operator used is a QKT with a stronger kick strength $\alpha' = 3.015$. Hence, the perturbation operator is $e^{-i(\delta J_z^2/2j\hbar)}$ where $\delta = \alpha' - \alpha$.

The classical kicked top is a map on the unit sphere, $x^2 + y^2 + z^2 = 1$:

$$x' = z, \qquad y' = x\sin(\alpha z) + y\cos(\alpha z),$$
$$z' = -x\cos(\alpha z) + y\sin(\alpha z). \tag{6}$$

For $\alpha = 3$ there are two fixed points of order one at the center of the regular regions of the map. They are located at

$$x_f = z_f = \pm 0.629\,412\,6, \qquad y_f = 0.455\,718\,7. \tag{7}$$

The regular regions of the classical phase space are seen clearly in Fig. 4.

To locate the edge of quantum chaos, we work in the oo (even under a 180° rotation about x and odd under a 180° about y; see p. 359 of [4]) symmetry subspace of the QKT with $j = 120$ (in the oo subspace $N = j/2$). We set y equal to y_f of the positive fixed point and change z so that the initial state can be systematically moved closer and further from the fixed point of the map. An initial state with a power-law decrease of overlap is found at $z_f - 0.124$. The overlap decay for this state at the edge of quantum chaos is illustrated in Fig. 2 and is very well fit by the solution of $dO/d(t^2) = -O^{q_{\rm rel}}/\tau^2_{q_{\rm rel}}$ ("rel" stands for relaxation). Although we do not know how to derive this differential equation from first principles, the numerical agreement is remarkable (see also [21]). A time-dependent q-exponential expression analogous to the one shown here has recently been proved for the edge of chaos and other critical points of the classical logistic map [19]. The polynomial overlap decay is the transition between the quadratic and exponential overlap decays. This transitory region does not appear for chaotic states (as shown in Fig. 1) and is a signature of the edge of quantum chaos.

A power law also emerges for the above initial state in the weak perturbation or Gaussian regime. Here we use $\alpha' = 3.0003$. The power law in this regime is illustrated in Fig. 2 and fit with the above equation.

The value of $q_{\rm rel}$ remains constant at 3.8 for small perturbations until the critical perturbation strength, δ_c, when the typical off diagonal elements of V are larger than Δ. We can approximate $\delta_c \simeq \sqrt{2\pi/N^3} = 5.4 \times 10^{-3}$ [12]. When δ is larger than δ_c, $q_{\rm rel}$ increases. The behavior of $q_{\rm rel}$ and $\tau_{q_{\rm rel}}$ versus δ can be seen in Fig. 3.

The location of the edge of quantum chaos for the QKT of spin 120 does not match up with the edge of the classical kicked top which appears at approximately $z_f - 0.2296$. This implies that classically regular regions of the kicked top appear chaotic on the QKT. As j is increased, the top becomes more and more classical and states exhibiting edge of quantum chaos behavior are centered closer to the classical value for the edge of chaos. Hence, for $j = 150$, 180, 210, and 240, the edge is observed at $z_f - 0.124$, 0.139, 0.151, 0.160, and 0.176, respectively.

The edge of chaos in the quantum and classical maps are not observed at the same value due to the size of the angular momentum coherent state. The coherent state

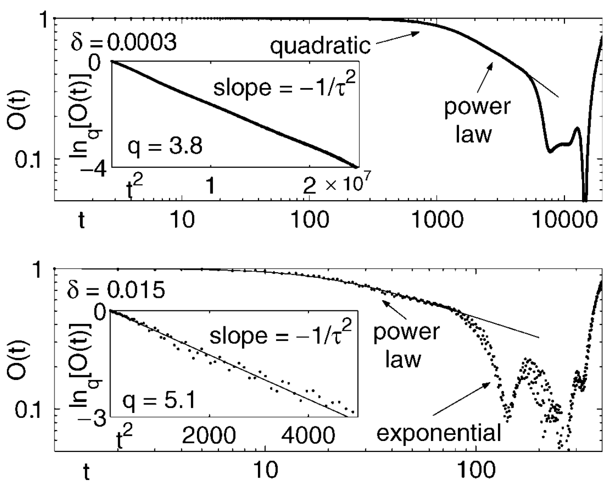

FIG. 2. Overlap versus time for an angular momentum coherent state initially located at the border between regular and chaotic zones of the QKT of spin 120 and $\alpha = 3$. This region is called the edge of quantum chaos and shows the expected power law decrease in overlap. The top figure is for a perturbation strength within the FGR regime, $\delta = 0.015$ and the bottom figure is for a perturbation strength of $\delta = 0.0003$, well below the FGR regime. On the log-log plot the power law decay region, from about 20–70 in the FGR regime and 2000–7000 in the Gaussian regime, is linear. We can fit the decrease in overlap with the expression $[1 + (q_{\rm rel} - 1)(t/\tau_{q_{\rm rel}})^2]^{1/(1-q_{\rm rel})}$ where, in the FGR regime, the entropic index $q_{\rm rel} = 5.1$ and $\tau_{q_{\rm rel}} = 40$ and in the Gaussian regime $q_{\rm rel} = 3.8$ and $\tau_{q_{\rm rel}} = 2500$. The insets of both figures show $\ln_{q_{\rm rel}} O \equiv (O^{1-q_{\rm rel}} - 1)/(1 - q_{\rm rel})$ versus t^2; since $\ln_q x$ is the inverse function of $e_q^x \equiv [1 + (1-q)\,x]^{1/(1-q)}$, this produces a straight line with a slope $-1/\tau^2$ (also plotted).

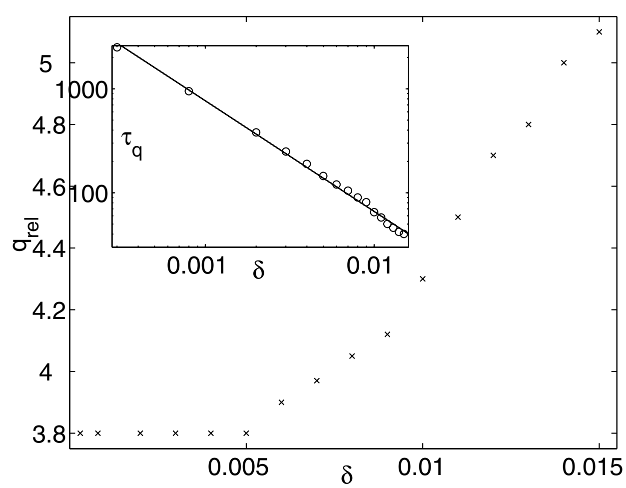

FIG. 3. $q_{\rm rel}$ versus perturbation strength. The value of $q_{\rm rel}$ remains constant at 3.8 until the critical perturbation, after which it increases. The relationship of $\tau_{q_{\rm rel}}$ versus perturbation strength is shown on a log-log plot in the inset and is well fit by a line with slope -1.06.

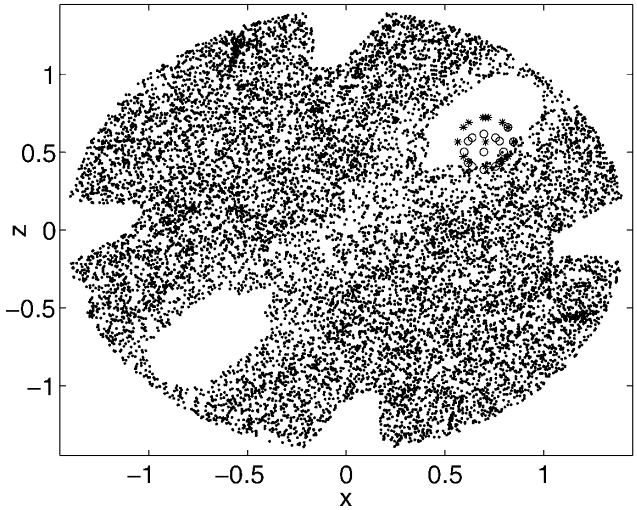

FIG. 4. Classical phase space of the kicked top with angular momentum coherent wave functions. 10 000 iterations of a chaotic orbit starting from the point $x = 0.6294$, $y = 0.7424$, $z = 0.2294$. The spherical phase space and the ellipsoidal coherent states are projected onto the x-z plane (only $y > 0$ shown) by multiplying the x and z coordinates of each point by R/r where $R = \sqrt{2(1 - |y|)}$ and $r = \sqrt{(1 - y^2)}$ [4]. The regular regions of the kicked top are clearly visible. Shown is a $j = 120$ wave function (stars) and a $j = 240$ (circles) wave function both at the edge of quantum chaos. Note that the variance of the $j = 120$ wave function is much larger than the variance of the $j = 240$ wave function. Hence, the behavior characteristic of the edge of quantum chaos appears further from the fixed point of the classical map.

grows as j decreases causing it to "leak out" into the chaotic region even though it is centered away from the chaotic border. This causes behavior characteristic of the edge of chaos to appear at different values depending on the dimension of Hilbert space. Figure 4 shows the wave functions for two values of j superimposed on the classical phase space. This gives an idea as to how large the wave function is compared to the regular region of the map.

In the region of j values that we have explored, no significant changes have been detected for q_{rel}, because the δ_c changes only slightly. However, $\tau_{q_{\mathrm{rel}}}$ decreases with increasing j.

To conclude, we have located a region on the border of chaotic and nonchaotic quantum dynamics. Quantum states located in this region exhibit a power-law decrease in overlap as opposed to the exponential overlap decay exhibited by fully chaotic quantum dynamics. The classical parallel to this region is the border between regular and chaotic classical dynamics which is characterized by the generalized Lyapunov coefficient.

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