

Faraday Patterns in Bose-Einstein Condensates

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Temporal periodic modulation of the interatomic s -wave scattering length in Bose-Einstein condensates is shown to excite subharmonic patterns in the atom density through a parametric resonance. The dominant wavelength of the spatial structures is primarily selected by the excitation frequency but also affected by the depth of the spatial modulation via a nonlinear resonance. These phenomena represent analogues of the Faraday patterns excited in vertically vibrated liquids.

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The very essence of spontaneous pattern formation is that a uniform state loses its stability against spatially modulated states when an external control parameter is varied. The dominant wavelength of the instability and the symmetries of the selected patterns are intrinsic properties of the system, independent of (or only weakly dependent on) initial or boundary conditions. Spontaneous pattern formation is widespread in natural systems and in laboratory experiments [1], however not yet suggested for the most recently created state of matter, the Bose-Einstein condensate (BEC) [2].

In this Letter, we report on the spontaneous symmetry breaking and the appearance of patterns and quasipatterns in a BEC subjected to a temporal modulation of the atomic s -wave scattering length. We show analytically and numerically that the dominant wave number of the patterns is selected by the excitation frequency through a dispersion-induced mechanism, and that the resulting patterns resemble those observed in the Faraday instability on a free surface of a fluid subject to oscillatory vertical acceleration [3]. We note that we deal with the periodic modulation of the *scattering length* of the condensate, in contrast to [4], where the periodic modulation of the *trap parameter* was investigated. Experimentally periodic modulation of the scattering length is achievable, e.g., via the Feshbach resonance [5], via the interaction with laser light [6], or via external dc electric fields [7]. In low dimensional condensates, such as the ones we consider here, the scattering length can also be tuned by changing the trap frequency corresponding to the tightly confined direction [8].

We consider a trapped BEC with atomic s -wave scattering length $a(t) = \bar{a}[1 + 2\alpha \cos(2\omega t)]$ periodically modulated in time around its mean value \bar{a} . We assume a pancake-type trapping potential $V_{\text{trap}}(x, y, z) = \frac{1}{2}\omega_z^2 z^2 + V(x, y)$, with $V(x, y) = \frac{1}{2}\omega_{\perp}^2(x^2 + y^2)$ and $\omega_z \gg \omega_{\perp}$, so that the BEC can be regarded as a two-dimensional (2D) system [9] extended in the weakly confining plane (x, y) . Patterns in such 2D system can be studied in the framework of the Gross-Pitaevskii (GP) equation [10,11] for the

BEC mean field $\psi(\mathbf{r}, t)$, $\mathbf{r} = (x, y)$, which, using suitably normalized variables, can be written as

$$i \frac{\partial \psi}{\partial t} = -\nabla^2 \psi + V(\mathbf{r})\psi + c(t)|\psi|^2 \psi. \quad (1)$$

Here $c(t) = a(t)/\bar{a} = 1 + 2\alpha \cos(2\omega t)$. In the absence of modulation ($\alpha = 0$) the BEC is assumed to be in its ground state $\psi(\mathbf{r}, t) = \psi_0(\mathbf{r})e^{-i\mu t}$, which can be computed numerically [11]. Since we study the pattern forming instabilities on a spatial scale much smaller than the size of the condensate, we consider the limiting case of a flat potential $V(\mathbf{r}) = 0$, which leads to $\mu = 1$ and $\psi_0(\mathbf{r}) = 1$ with a proper scaling of space and time in Eq. (1). We show below numerically that the pattern forming instability found in the flat trapping potential persists in harmonic trapping potentials.

The spatially homogeneous, temporally periodic solution of Eq. (1) in the flat potential limit is $\psi(\mathbf{r}, t) = \psi_{\text{hom}}(t) \equiv \exp[-it - i\frac{\alpha}{\omega} \sin(2\omega t)]$. In order to determine whether the *spatially uniform* periodic modulation can induce a spontaneous spatial-symmetry breaking of this homogeneous state, we perform a linear stability analysis against *spatially modulated* perturbations. Setting in Eq. (1) $\psi(\mathbf{r}, t) = \psi_{\text{hom}}(t)[1 + w(t)\cos(\mathbf{k} \cdot \mathbf{r})]$, where $w(t)$ is the complex-valued amplitude of the perturbation, one obtains for $u = \text{Re}(w)$ the following Mathieu equation:

$$d^2 u / dt^2 + [\Omega^2(k) + 4k^2 \alpha \cos(2\omega t)]u = 0, \quad (2)$$

where $\Omega(k) = k\sqrt{k^2 + 2}$ is the dispersion relation of the perturbations in the absence of driving and $k^2 = \mathbf{k} \cdot \mathbf{k}$. The Floquet exponents $\sigma = \sigma(k, \omega, \alpha)$ for Eq. (2) describe the stability of the homogeneous BEC state. An instability with wave number k arises on the neutral stability curve $\alpha = \alpha_N(k, \omega)$, implicitly defined by the relation $\text{Re}\sigma(k, \omega, \alpha) = 0$, and the homogeneous state is unstable if $\text{Re}\sigma > 0$. The neutral stability curve of Eq. (2) is composed of an infinite series of resonance tongues located around the wave numbers $k = k_n \equiv \sqrt{-1 + \sqrt{1 + (n\omega)^2}}$, n integer, which are selected by the

parametric resonance condition $\Omega(k_n) = n\omega$ between the external forcing frequency 2ω and the natural frequency Ω of the system, Fig. 1(a). This wave number selection is analogous to the one generally found in parametrically forced spatially extended systems, such as in the Faraday instability of vertically vibrated fluids [12].

Numerical simulations for the flat potential were performed to test the parametric instability of the homogeneous BEC state. Equation (1) was integrated using a pseudospectral split-step technique in a square spatial domain with periodic boundary conditions. Figure 2 shows snapshots of the spatial atom density $|\psi(\mathbf{r}, t)|^2$ (upper row) and of the atom momentum density $|\tilde{\psi}(\mathbf{k}, t)|^2$ (lower row) where the tilde denotes spatial Fourier transformation. The figure shows the growth of resonant modes, located on concentric rings in momentum space in agreement with the linear stability analysis. First the main resonant ring in momentum space appears, which corresponds to transient quasipatterns in physical space, see Fig. 2(a). Further, see Fig. 2(b), higher-order resonance rings appear in momentum space, which correspond to the higher-order resonance tongues of Fig. 1(a). These structures appear transiently due to the conservative nature of the GP Eq. (1). For a long time one observes heating and eventual destruction of the condensate, Fig. 2(c). This destruction, which we observe in all our 2D numerics, contrasts with the everlasting periodic revivals of the spatial modulation in the 1D case [13].

In order to determine the intrinsic symmetries of the parametric patterns, we investigated weakly dissipative BECs, since the inclusion of dissipative terms, which describe damping mechanisms of trapped BECs, can be expected to lead to the selection of stationary patterns with a well-defined symmetry. We adopt a phenomenological description of dissipation [14], by including damping in Eq. (1) and obtain the parametrically driven,

damped GP equation,

$$i \frac{\partial \psi}{\partial t} = (1 - i\gamma)(-\nabla^2 - \mu + |\psi|^2)\psi + 2\alpha \cos(2\omega t)|\psi|^2\psi, \quad (3)$$

where $\mu = 1$ is the chemical potential. The damping, described by the adimensional coefficient γ , ensures an evolution towards the ground state in the absence of parametric driving ($\alpha = 0$). Also the damping sets a nonzero threshold value for the parametric instability, and removes the degeneracy of threshold values for higher resonance tongues. Analogously to Eq. (2), we find in the dissipative case a damped Mathieu equation:

$$d^2u/dt^2 + 2\gamma(1 + k^2)du/dt + [(1 + \gamma^2)\Omega^2(k) + 4k^2\alpha \cos(2\omega t)]u = 0. \quad (4)$$

The neutral stability curves obtained from a Floquet analysis of Eq. (4) are shown in Fig. 1(b). An approximate expression for the neutral stability curve of the first resonance tongue, as calculated by a perturbative analysis, is

$$\alpha_N(k, \omega) = \frac{\sqrt{2 + k^2}}{k} \sqrt{(\omega - \Omega)^2 + \gamma^2(1 + k^2)^2}. \quad (5)$$

Numerical integration of Eq. (3) with small dissipation $\gamma = 0.03$ shows the formation of stationary spatial patterns with different symmetries, as shown in Fig. 3. In momentum space, only several modes corresponding to the first resonance tongue survive due to nonlinear competition. The pattern symmetries depend on the excitation frequency: for large modulation frequencies typical patterns are squares, Fig. 3(a), or quasiperiodic patterns with eightfold symmetry, Fig. 3(b). For moderate frequencies

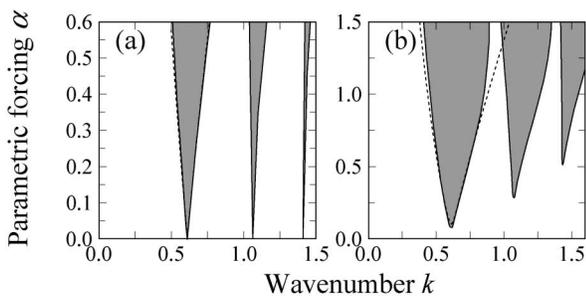


FIG. 1. Resonance tongues of the parametric instability for the conservative $\gamma = 0$ (a) and dissipative $\gamma = 0.03$ (b) GP equation for $\omega = 0.3\pi$. Shaded domains indicate the regions where the uniform condensate state is unstable, as following from Floquet analysis of Eqs. (2) and (4). The dashed curves show the neutral stability curve for the first resonance tongue as given by Eq. (5). Notice that the atomic interaction remains repulsive for $\alpha < 0.5$, and we will limit our analysis to this regime.

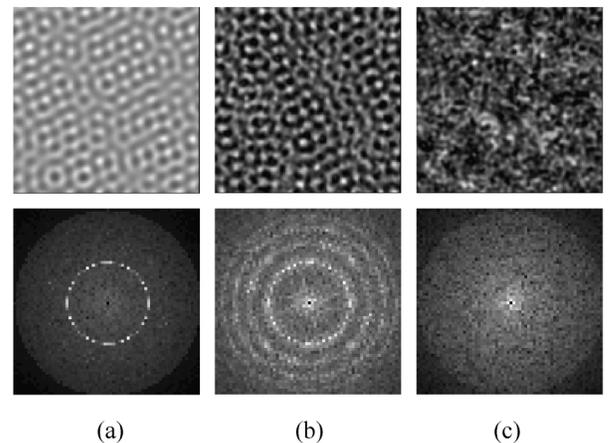


FIG. 2. Evolution of patterns in parametrically driven BECs, as obtained by numerical integration of Eq. (1) in a flat potential with periodic boundary conditions, for $\alpha = 0.2$ and $\omega = 1.5\pi$. Upper row: distribution in physical space; bottom row: distributions in momentum space. The snapshots were taken at times: (a) $t = 100$, (b) $t = 200$, (c) $t = 300$. The zero component in momentum space pictures is removed.

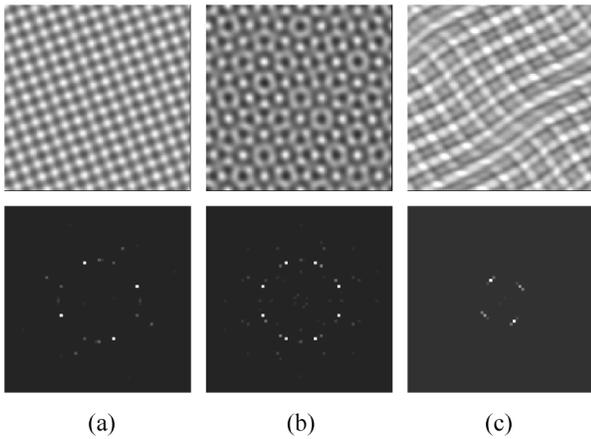


FIG. 3. Patterns in dissipative BECs, as obtained by numerical integration of Eq. (3) in a flat potential with periodic boundary conditions for $\alpha = 0.2$, $\gamma = 0.03$, and (a) $\omega = 1.5\pi$, (b) $\omega = 1.5\pi$, and (c) $\omega = 0.5\pi$. Upper row: distribution in physical space; bottom row: distributions in momentum space. The zero component in momentum space pictures is removed.

rhombic patterns are favored, Fig. 3(c). These results have been independently tested by direct integration of a set of amplitude equations derived from Eq. (3) using a Galerkin decomposition (see, e.g., [15]). We note that the symmetries of the observed patterns are those matching the parametric resonance condition $\Omega(k_n) = n\omega$ via the following scenario: (i) In the linear stage of the instability a set of modes with wave vector $\{\mathbf{k}^{(i)}, -\mathbf{k}^{(i)}\}_{i=1,2,\dots}$ ($|\mathbf{k}^{(i)}| = k_1$) lying on the first resonant ring are excited; (ii) Owing to nonlinear interactions, passive modes with wave vector $\mathbf{k}^{(i,j)} = \mathbf{k}^{(i)} + \mathbf{k}^{(j)}$ are subsequently excited, with frequency 2ω . Among these wave vectors are some matching the parametric resonance condition associated with the second ring ($|\mathbf{k}^{(i,j)}| = k_2$). In the case of patterns composed of only two fundamental wave vectors, e.g., $\pm\mathbf{k}^{(1)}$ and $\pm\mathbf{k}^{(2)}$ with an angle θ between them, the resonance condition $|\mathbf{k}^{(1,2)}| = k_2$ is $\theta = 2 \arccos(k_2/2k_1)$. For small driving frequencies ($\omega^2 \ll 1$), $\theta = \sqrt{3}\omega + O(\omega^3)$, and for large ω , $\theta = \pi/2 + O(1/\omega)$. These considerations make our observations of square patterns at large driving frequencies, and of rhombic patterns for moderate and small frequencies (Fig. 3) plausible. Also, we observe that the dominant angle of the rhombic pattern is dependent on the excitation frequency. However, we were unable to ascertain definitely the above relations between the frequency of excitation and the angle: the local angle of the rhombic pattern generally varies in space, Fig. 3(c), and also in time, thus it is not strictly defined.

The difficulties in determining the symmetries of the patterns occur possibly due to a nonlinear spatial resonance. To give analytical evidence of this nonlinear resonance, an amplitude equation for the simplest pattern

(a nearly resonant, weakly nonlinear roll) was derived by a standard multiple scale expansion (for details, see [16]). We expand the condensate mean field $\psi(x, t) = \psi_{\text{hom}}(t)[1 + w(t) \cos(kx) + \sum_{n=2}^{\infty} \varepsilon^n w_n(x, t)]$, assume that the modulation depth of the pattern is small, $w(t) = O(\varepsilon)$, ε being a small parameter, and use the following parameter scalings, as suggested by the linear stability analysis: $\omega = \Omega(k) + O(\varepsilon^2)$, $\gamma = O(\varepsilon^2)$, and $\alpha = O(\varepsilon^2)$. A slow time $\tau = \varepsilon^2 t$ is defined and all coefficients of the expansion are assumed to depend on both t and τ ($\partial_t \rightarrow \partial_t + \varepsilon^2 \partial_\tau$). Substituting the above expansion and scalings into Eq. (3), and equating equal powers in ε , at leading order one has $w(t) = (1 - \Omega/k^2)R(t)e^{i\omega t} + (1 + \Omega/k^2)R^*(t)e^{-i\omega t}$, where the complex amplitude R satisfies a complex Landau equation with broken phase symmetry:

$$dR/dt = -(c_1 + ic_2)R + ic_3R^* - ic_4|R|^2R, \quad (6)$$

where $c_1 = \gamma(1 + k^2)$, $c_2 = (\omega - \Omega)$, $c_3 = \alpha k^2/\Omega$, and $c_4 = (3 + 5k^2)/\Omega$. A relevant measurable quantity is the time averaged occupation of the nonhomogeneous (excited) modes of the condensate which, at leading order, is $\rho_{\text{mod}} = 2\langle |\frac{1}{2}w(t)|^2 \rangle = (1 + \Omega^2/k^4)|R|^2$. The steady solution ($dR/dt = 0$) of Eq. (6) results in:

$$\rho_{\text{mod}} = \left(1 + \frac{\Omega^2}{k^4}\right) \times \frac{-\Omega(\omega - \Omega) \pm \sqrt{(\alpha k^2)^2 - [\gamma\Omega(1 + k^2)]^2}}{3 + 5k^2}. \quad (7)$$

which shows the occurrence of the nonlinear resonance. We tested (7) by a numerical analysis of Eq. (3) in 1D using a Galerkin expansion $\psi(x, t) = \psi_{\text{hom}}(t) \times \sum_{n=-M}^M g_n(t)e^{inkx}$, with k fixed and M a truncation index,

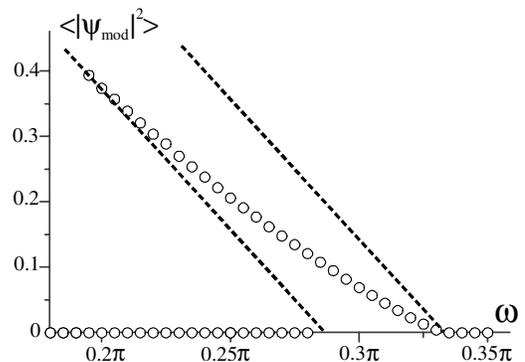


FIG. 4. Intensity of nonhomogeneous (modulated) spatial modes in dissipative BECs, as obtained by numerical integration of Eq. (3) in a Galerkin mode expansion. The eigenfrequency of the lowest modulated mode $k = 0.1 \times 2\pi$ is $\Omega(k) = 0.3095\pi$. The parameters are $\gamma = 0.05$ and $\alpha = 0.25$. The dashed lines are given by Eq. (7). Note the bistability between the homogeneous state (lower branch of circles), and modulated state (upper branch) in some region of modulation frequency (between $\omega = 0.2\pi$ and $\omega = 2.8\pi$ in this case).

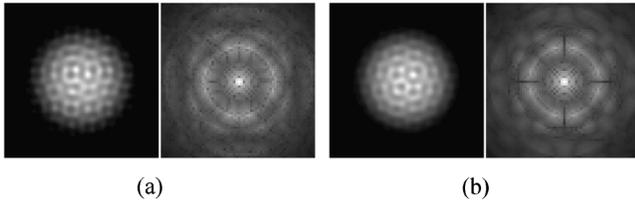


FIG. 5. Density distributions of trapped BECs, as obtained by numerical integration of Eq. (3), in a parabolic potential. Snapshot of a transient pattern, in physical space (left) and in momentum space (right): (a) in a conservative BEC ($\gamma = 0$); (b) in a dissipative BEC ($\gamma = 0.03$). Parameters: $\omega = 1.5\pi$ and $\alpha = 0.2$ ($\alpha = 0.4$) for conservative (dissipative) case. The trap frequency at these conditions was equal to $\omega_{\text{trap}} = 0.08\pi$. The horizontal and vertical dark lines in momentum space are the artifacts of numerical grid.

and calculated the time averaged intensity of the spatial modulation $\rho_{\text{mod}} = \sum_{n \neq 0} \langle |g_n(t)|^2 \rangle$ as a function of the excitation frequency. Figure 4 shows the calculated nonlinear resonance: the dependence of the eigenfrequency of the spatial mode on the modulation depth is apparent, and corresponds well with that given by Eq. (7) (dashed lines). In the 2D case this nonlinear resonance is expected to complicate pattern selection since the symmetry becomes dependent not only on the driving frequency but also on the excitation amplitude through the spatial modulation depth of the condensate density.

Finally, numerical simulations were performed for the case of 2D condensates with a harmonic potential trap (Fig. 5) in order to show the persistence of the pattern forming instability under experimentally relevant conditions. The dominant wavelength, as calculated from the momentum distributions, corresponds well to the analytical value predicted by Floquet analysis of Eqs. (2) and (4). In physical units, the simulation corresponds to a density $N = 10^5$ of ^{87}Rb atoms in a magnetic trap of frequency $\omega_{\perp} = 2\pi \times 10 \text{ s}^{-1}$. This results in a condensate size (diameter at half the density maximum) equal to $50 \mu\text{m}$ with a mean scattering length $\bar{a} = 5.2 \text{ nm}$.

In conclusion, we have shown that spontaneous pattern formation, a very general phenomenon studied in different fields of nonlinear science, occurs also for the newly created state of matter, the BECs. Pattern formation can be achieved by modulation of the atomic scattering length, a mechanism that bears a close connection with the formation of spatial patterns on the surface of a vibrating fluid. We studied 2D condensates, in which the dynamics occurs in the plane transverse to the tight confinement direction, and found squares, rhombi, and octagons as typical patterns. We also revealed the existence of a nonlinear spatial resonance, a phenomenon which may

lead to the expectation of bistable localized structures, such as those found, e.g., in vibrated fluids [17] or in nonlinear optical resonators [18]. We thus envisage that such localized structures can be excited in parametrically driven BECs by periodic modulation of the interatomic scattering length.

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