Effective Size of Certain Macroscopic Quantum Superpositions

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Several experiments and experimental proposals for the production of macroscopic superpositions naturally lead to states of the general form $|\phi_1\rangle^{\otimes N} + |\phi_2\rangle^{\otimes N}$, where the number of subsystems N is very large, but the states of the individual subsystems have large overlap, $|\langle\phi_1|\phi_2\rangle|^2 = 1 - \epsilon^2$. We propose two different methods for assigning an effective particle number to such states, using ideal Greenberger-Horne-Zeilinger states of the form $|0\rangle^{\otimes n} + |1\rangle^{\otimes n}$ as a standard of comparison. The two methods are based on decoherence and on a distillation protocol, respectively. Both lead to an effective size n of the order of $N\epsilon^2$.

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It was pointed out already in 1935 by Schrödinger [1] that quantum mechanics predicts the existence of superpositions of macroscopically distinct states. The observation of the related quantum interference effects is very difficult because of environment-induced decoherence. Nevertheless several methods for producing and verifying macroscopic superpositions have been proposed, in systems ranging from superconductors [2] over Bose-Einstein condensates (BECs) [3,4] and optomechanical systems [5] to small cantilevers coupled to superconducting islands [6]. Recently there have even been the first experimental demonstrations of the superposition of distinct macroscopic current states in superconducting quantum interference devices (SQUIDs) [7].

The states produced in such proposed experiments can often be described to a good approximation by

$$|\psi\rangle = \frac{1}{\sqrt{K}}(|\phi_1\rangle^{\otimes N} + |\phi_2\rangle^{\otimes N}), \tag{1}$$

with $K = 2 + \langle\phi_1 \mid \phi_2\rangle^N + \langle\phi_2 \mid \phi_1\rangle^N$. Here $|\psi\rangle$ is a state of N two-level systems (qubits). The individual qubits could be seen as simple models for many different physical systems, including the atoms in a BEC inside a double-well potential [3], atoms in two internal states, or the Cooper pairs in a SQUID (which can flow in a clockwise or an anticlockwise direction).

The essential point for our present discussion is that the states $|\phi_1\rangle$ and $|\phi_2\rangle$ are not necessarily orthogonal. In fact, we will study the case where $|\langle\phi_1 \mid \phi_2\rangle|^2 = 1 - \epsilon^2$ is very close to 1 (ϵ is small). Note that in spite of this the overlap between the two terms in (1) can be very small for large N, since it is given by $|\langle\phi_1 \mid \phi_2\rangle|^{2N} = (1 - \epsilon^2)^N$, which is well approximated by $e^{-N\epsilon^2}$ for small ϵ.

We investigate how states of the form (1) compare to ideal Greenberger-Horne-Zeilinger (GHZ) states of the form

$$|\mathrm{GHZ_n}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}). \tag{2}$$

States of the form (2) can be produced to good approximation in quantum optical systems, including atoms in cavity quantum electrodynamics [8], trapped ions [9], and photons from parametric downconversion [10]. So far, particle numbers n up to 4 have been achieved. On the other hand, states of the form (1) can involve macroscopic numbers of particles, in combination with small values of ϵ. An important question is whether there is a well-defined way to compare these two—very different—cases. That is, can one give a meaningful answer to the question whether, e.g., a state of the form (1) with $N = 10^6$ particles, but $\epsilon = 10^{-3}$ is more or less entangled than an ideal GHZ state with $n = 10$?

A first simple way of assessing the "size" of states of the form (1) is to look at the overlap between the two terms. However, as pointed out above, this will be close to zero in most interesting cases and thus does not lead to a very sensitive criterion. A very intuitive way of comparing (1) and (2) would be to assign to (1) an "effective particle number" n, which could be interpreted as saying that the state (1) is (in a certain well-defined sense) *equivalent* to an ideal GHZ state of n qubits. This requires well-defined and physically meaningful methods of determining such an effective particle number. Here we will propose two such methods and show that they lead to essentially the same result, namely, that the effective n of a state of the form (1) is of the order of $N\epsilon^2$.

Our two methods are very different. The first one is based on the rate of decoherence. An important potential application of states of the form (1) is for the observation of (or the search for) weak decoherence processes, ultimately including unconventional ones as predicted by spontaneous wave-function collapse models [11]. If the state (1) is as sensitive to decoherence as an ideal n-qubit GHZ state, it is natural to say that its effective size is n.

The second method of assigning an effective size to our states is in the spirit of quantum information, viewing multiparty entanglement as a convertible resource. We ask how much ideal GHZ entanglement can be *distilled* from the states (1) by local operations (acting separately on each qubit) and classical communication.

It is remarkable that these two very different approaches lead to the same result. We believe that this suggests that $N\epsilon^2$ is indeed a good physical quantification of the size of macroscopic superpositions of the present type. We conclude by giving another simple argument in favor of the proposed scaling of the effective size, based on particle loss.

Let us now follow our first approach and study the effect of local decoherence on the state (1). We will consider the case where each of the particles undergoes an independent decoherence process, i.e., each particle is coupled to an independent bath. The effect of decoherence is quantified as the rate of decay of the off-diagonal elements in the natural basis. Note that we are interested in properties of states and not of physical setups which generate them. In this sense, although for different physical systems the decoherence process may be completely different, we can study the behavior of the states describing those systems under a certain decoherence process in order to compare the properties of these states.

We consider phase decoherence in the natural basis; that is diagonal elements in the $\{|0\rangle, |1\rangle\}$ basis remain unchanged, while off-diagonal elements $|0\rangle\langle 1|, |1\rangle\langle 0|$ decay with a rate γ. The decoherence process of an individual system is described by $|i\rangle\langle j| \to e^{-\gamma t}|i\rangle\langle j|$ for $i \neq j$ and $|j\rangle\langle j| \to |j\rangle\langle j|$, which corresponds—in a quantum information language—to a dephasing channel. The action of this channel may be described by the completely positive map $\mathcal{E}$ defined through $\mathcal{E}(\rho) = p_0\rho + (1-p_0)\sigma_z\rho\sigma_z$, where $p_0 = (1 + e^{-\gamma t})/2$ and σ_z is a Pauli matrix.

It is straightforward to establish the effect of this decoherence process on an ideal n-particle GHZ state. The density matrix for the state Eq. (2) is $1/2(|0\rangle\langle 0|^{\otimes n} + |1\rangle\langle 1|^{\otimes n} + |0\rangle\langle 1|^{\otimes n} + |1\rangle\langle 0|^{\otimes n})$. Since $\mathcal{E}^{\otimes n}(\sigma^{\otimes n}) = [\mathcal{E}(\sigma)]^{\otimes n}$, one can study the decay of the off-diagonal elements $|0\rangle\langle 1|^{\otimes n}$ and $|1\rangle\langle 0|^{\otimes n}$ by considering the action of the decoherence channel $\mathcal{E}$ on the single-qubit operators $|0\rangle\langle 1|$ and $|1\rangle\langle 0|$. To be specific, we consider the trace norm, $\|A\|_1 \equiv \mathrm{tr}\sqrt{A^\dagger A}$, of $a_t = \mathcal{E}(a_0)$, where $a_0 = |0\rangle\langle 1|$. Note that $\|A^{\otimes N}\|_1 = \|A\|_1^N$. Since $a_t = e^{-\gamma t}a_0$, we have that $\|a_t^{\otimes n}\|_1 = \|a_t\|_1^n = e^{-\gamma n t}\|a_0\|_1$. The off-diagonal element of the GHZ state decays with a rate γn.

We want to compare this to the decay rate of the off-diagonal terms for the N-particle states $|\psi\rangle$ of the form (1). Without loss of generality we denote

$$|\phi_1\rangle = |0\rangle, \qquad |\phi_2\rangle = \cos(\epsilon)|0\rangle + \sin(\epsilon)|1\rangle, \quad (3)$$

and use the shorthand notation $c_\epsilon \equiv \cos(\epsilon)$ and $s_\epsilon \equiv \sin(\epsilon)$. We have that $K = 2(1 + c_\epsilon^N)$ and for small ϵ, $|\langle\phi_1 \mid \phi_2\rangle|^2 = c_\epsilon^2 \approx 1 - \epsilon^2$. The density matrix for the state Eq. (1) is $1/K(|\phi_1\rangle\langle\phi_1|^{\otimes N} + |\phi_2\rangle\langle\phi_2|^{\otimes N} + |\phi_1\rangle \times \langle\phi_2|^{\otimes N} + |\phi_2\rangle\langle\phi_1|^{\otimes N})$. We are interested in the decay rate of the off-diagonal elements. As before, the problem can be reduced to studying single-qubit operators, namely, $b_0 \equiv |\phi_1\rangle\langle\phi_2| = c_\epsilon|0\rangle\langle 0| + s_\epsilon|0\rangle\langle 1|$. We have that b_0 changes due to the above decoherence process to $b_t = \mathcal{E}(b_0) = \sqrt{d}|0\rangle\langle\chi_t|$ with $|\chi_t\rangle = 1/\sqrt{d}(c_\epsilon|0\rangle + s_\epsilon e^{-\gamma t}|1\rangle)$ and $d = c_\epsilon^2 + s_\epsilon^2 e^{-2\gamma t}$, such that $|\chi_t\rangle$ is properly normalized. It follows that $\|b_t^{\otimes N}\|_1 = d^{N/2}$. For $\epsilon \ll 1$, $t \ll \gamma^{-1}$, $N\epsilon^2\gamma t \ll 1$, we have $d \approx 1 - 2\epsilon^2\gamma t$ and thus $d^{N/2} \approx e^{-\gamma N\epsilon^2 t}$. This implies that the rate with which the coherences of the state $|\psi\rangle$ decay is given by $\gamma N\epsilon^2$. That is, the decoherence rate of a state of the form Eq. (1) with $|\langle\phi_1 \mid \phi_2\rangle|^2 = 1 - \epsilon^2$ is the same as that of an ideal n-party GHZ state with $n = N\epsilon^2$ and thus one may associate an effective particle number $n = N\epsilon^2$ to the state $|\psi\rangle$.

The observed decoherence rate is not restricted to this specific decoherence model. Consider, for example, the basis independent decoherence model of a partially depolarizing channel. In this case, the individual decoherence process for each qubit is described by $|i\rangle\langle j| \to \mu|i\rangle\langle j| + (1-\mu)\delta_{i,j}\frac{1}{2}\mathbb{1}$ where $\mu \equiv e^{-\gamma t}$. Equivalently, the completely positive map $\tilde{\mathcal{E}}$ describing this process is given by $\tilde{\mathcal{E}}(\rho) = \sum_{i=0}^3 p_i\sigma_i\rho\sigma_i$ with $p_0 = (3\mu + 1)/4$ and $p_1 = p_2 = p_3 = (1-\mu)/4$, where $\sigma_0 = \mathbb{1}$, and the σ_i are Pauli matrices. We find that a_0 $[b_0]$ changes due to this decoherence process to $a_t = \tilde{\mathcal{E}}(a_0) = \mu a_0$ $[b_t = \tilde{\mathcal{E}}(b_0) = c_\epsilon(1+\mu)/2|0\rangle\langle 0| + (1-\mu)/2|1\rangle\langle 1| + s_\epsilon\mu|0\rangle\langle 1|]$. One obtains that $\|a_t\|_1 = \mu = e^{-\gamma t}$ and $\|b_t\|_1 = \sqrt{c_\epsilon^2 + \mu^2 s_\epsilon^2} = \sqrt{d}$, which is exactly the same as in the case of the dephasing channel. One thus recovers exactly the same decoherence rates, γn and $\gamma N\epsilon^2$, respectively, as in the case of the dephasing channel.

Let us now turn to our second approach, which is more in the spirit of quantum information. We again consider states $|\psi\rangle$ of the form (1) with $|\phi_{1,2}\rangle$ defined in Eq. (3). We are interested in the *distillation* of ideal n-particle GHZ states (2) from these states under the condition that only local operations and classical communication are allowed. The restriction to local operations is essential if one wants to quantify the entanglement contained in a given state because nonlocal operations could create additional entanglement. We are interested only in the number of particles which form a GHZ state after the distillation process, i.e., the effective size of the GHZ state, and not which of the particles is entangled.

We show that the average number of the particles which is in an ideal GHZ state after the distillation process scales essentially as $n = N\epsilon^2$. We (i) provide an explicit protocol to produce—with unit probability—n-party GHZ states from a single copy of $|\psi\rangle$, where the average value of n is $N\epsilon^2/2$ and (ii) show that even in the asymptotic limit, i.e., considering several identical copies of the state $|\psi\rangle$, this average value is bounded from above by $n \approx N\epsilon^2[-\log_2(\epsilon)/2]$ [12].

Let us start with (i), a practical protocol which transforms a single copy of $|\psi\rangle$ deterministically into n-party GHZ states by means of local filtering measurements.

The protocol we propose is a generalization to multipartite systems of the distillation protocol of Ref. [13] for the optimal distillation of tripartite GHZ states from a single copy of an arbitrary pure state of three qubits. Consider the local filtering measurement described by the operator $A \equiv k(|0\rangle\langle\tilde{\phi}_1| + |1\rangle\langle\tilde{\phi}_2|)$, where $\{|\tilde{\phi}_1\rangle, |\tilde{\phi}_2\rangle\}$ is the biorthonormal basis to $\{|\phi_1\rangle, |\phi_2\rangle\}$, i.e., $\langle\phi_j \mid \tilde{\phi}_l\rangle = \delta_{jl}$. The constant k is chosen such that the other operator $\bar{A}$ of the local, two-outcome generalized measurement $\{A, \bar{A}\}$, which fulfills $A^\dagger A + \bar{A}^\dagger\bar{A} = \mathbb{1}$, has rank one. This implies that in case one obtains the outcome corresponding to A, then $A \otimes \mathbb{1}^{\otimes N-1}|\psi\rangle = k/\sqrt{K}(|0\rangle|\phi_1\rangle^{\otimes N-1} + |1\rangle|\phi_2\rangle^{\otimes N-1})$, while for the other outcome $\bar{A} \otimes \mathbb{1}^{\otimes N-1}|\psi\rangle \propto |\chi\rangle \otimes (|\phi_1\rangle^{\otimes N-1} + |\phi_2\rangle^{\otimes N-1})$, i.e., the measured particle factors out. The distillation protocol works as follows: Each of the parties performs locally the two-outcome generalized measurement $\{A, \bar{A}\}$ and all those n parties which obtained a positive outcome, i.e., the outcome corresponding to A, finally share an ideal n-party GHZ state of the form (2), while the remaining parties are in a product state. We shall be interested in the expectation value of n.

Using the notation of Eqs. (1) and (3), we find that

$$A = \frac{\sqrt{1 - c_\epsilon}}{s_\epsilon}\begin{pmatrix} s_\epsilon & -c_\epsilon \\ 0 & 1 \end{pmatrix}. \tag{4}$$

In the jth measurement, the probability to obtain the outcome corresponding to A is given by $p = (1 - c_\epsilon)/(1 + c_\epsilon^{N-j+1})$ provided that none of the previous measurements was successful. In case one of the previous measurements was already successful, the probability to obtain an outcome corresponding to A is given by $\tilde{p} = (1 - c_\epsilon)$ for the remaining parties. This different behavior after the first successful measurement can be easily understood by noting that once one of the measurements was successful, then the (normalized) state after the measurement is given by $1/\sqrt{2}(|0\rangle|\phi_1\rangle^{\otimes N'} + |1\rangle|\phi_2\rangle^{\otimes N'})$, while otherwise the normalization constant is $K = 2(1 + c_\epsilon^{N-j+1})$. One finds that the probability q_n to obtain n (where $n \geq 1$) successful measurements—and thus n-party GHZ states—is given by

$$q_n = (1 - c_\epsilon)^n c_\epsilon^{N-n}\binom{N}{n}\frac{1}{1 + c_\epsilon^N}, \tag{5}$$

while $q_0 = 2c_\epsilon^N/(1 + c_\epsilon^N)$. Note that this probability distribution is, up to the factor $1/(1 + c_\epsilon^N)$ and correspondingly the value of q_0, very similar to a binomial distribution. The expectation value $\langle n\rangle \equiv \sum_{j=0}^N q_j j$ is given by $\langle n\rangle = (1 - c_\epsilon)N/(1 + c_\epsilon^N)$ which simplifies for $\epsilon \ll 1$ and $N\epsilon^2 \gg 1$ to

$$\langle n\rangle \approx N\epsilon^2/2. \tag{6}$$

This provides the desired lower bound for the distillation rates of n-party GHZ states.

Regarding (ii), the announced upper bound for the distillation rate, we use the fact that the von Neumann entropy of the reduced density operator with respect to system 1 ρ_1 [14], $S_1(\rho_1) \equiv -\mathrm{tr}(\rho_1 \log_2 \rho_1)$, is an entanglement monotone, i.e., not increasing under local operations and classical communication [15–17].

We consider the distillation process in the asymptotic limit, i.e., the transformation of $M \to \infty$ identical copies of the state $|\psi\rangle$ to n-particle GHZ states [16]. Such a distillation protocol consists of an arbitrary sequence of local operations (including measurements), possibly assisted by classical communication, mathematically described by a multilocal superoperator [16]. The protocol produces a certain number, say M_n copies, of n-party GHZ states, which can, as we are interested only in the number of parties which constitute a GHZ state, without loss of generality be considered to be symmetrically distributed among the N parties [18]. Such a symmetric configuration is denoted by $|\overline{\mathrm{GHZ}}_n\rangle^{\otimes M_n}$ [19]. The distillation protocol is described by the following transformation:

$$|\psi\rangle^{\otimes M} \to \bigotimes_{n=2}^{N} |\overline{\mathrm{GHZ}}_n\rangle^{\otimes p_n M}, \tag{7}$$

where $p_n \geq 0$ denotes the average number of n-party GHZ states which are produced per copy from $|\psi\rangle$.

Given the monotonicity of the entropy under local operations, we obtain

$$MS_1(|\psi\rangle) \geq \sum_{n=2}^{N} S_1(|\overline{\mathrm{GHZ}}_n\rangle^{\otimes p_n M}), \tag{8}$$

where $S_1(|\overline{\mathrm{GHZ}}_n\rangle^{\otimes p_n M})$ denotes the entropy of the reduced density operator with respect to system 1 of $p_n M$ copies of (symmetrically distributed) n-particle GHZ states [19]. Since the probability that the first particle belongs to the n entangled particles is given by $p = \binom{N-1}{n-1}/\binom{N}{n} = n/N$, we have that $S_1(|\overline{\mathrm{GHZ}}_n\rangle^{\otimes p_n M}) = p_n M n/N$ and thus $\sum_{n=2}^N p_n n \leq N S_1(|\psi\rangle)$. It is straightforward to calculate $S_1(|\psi\rangle)$ using that for $|\psi\rangle$ (1), the reduced density operator with respect to system 1 is given by $\rho_1 = [(1 + c_\epsilon^2 + 2c_\epsilon^N)|0\rangle\langle 0| + s_\epsilon c_\epsilon(1 + c_\epsilon^{N-2})(|0\rangle\langle 1| + |1\rangle\langle 0|) + s_\epsilon^2|1\rangle\langle 1|]1/(2 + 2c_\epsilon^N)$. For $\epsilon \ll 1$ and $N\epsilon^2 \gg 1$, one obtains that $S_1(|\psi\rangle) \approx -\epsilon^2\log_2(\epsilon)/2$ which implies

$$\sum_{n=2}^{N} p_n n \leq -N\epsilon^2\frac{1}{2}\log_2(\epsilon) \tag{9}$$

as announced [20].

We would finally like to mention another simple argument that suggests the same scaling for the effective size of the states Eq. (1). Let us compare the effects of particle loss on the state (1) and on an ideal GHZ state. Suppose that every qubit is lost with a probability λ. Consider an n-qubit GHZ state. As soon as a single qubit is lost, the state becomes completely diagonal. There is only an off-diagonal element in the case of no losses, which has a probability of $(1 - \lambda)^n$. The expectation value of the off-diagonal element in the case of losses is therefore $\frac{1}{2}(1 - \lambda)^n$, equal to $\frac{1}{2}e^{-\lambda n}$ for small λ.

On the other hand, for the state (1), tracing out particles reduces the size of the off-diagonal terms but does not completely remove them. Tracing out k particles multiplies the off-diagonal terms by a factor of $\langle\phi_1|\phi_2\rangle^k = (1-\epsilon^2/2)^k$, equal to $e^{-k\epsilon^2/2}$ for small ϵ. The typical number of particles lost will be $N\lambda$, therefore the typical off-diagonal term will go like $e^{-\lambda N\epsilon^2/2}$. For large $N\lambda$ the probability distribution will be strongly peaked around the typical value. Therefore the expectation value of the off-diagonal term will be of order $e^{-\lambda N\epsilon^2/2}$. One sees that if the ideal GHZ state has $n = N\epsilon^2/2$, then the expectation values of the off-diagonal terms will have the same size for the two states. This is one more confirmation for our proposal that the "effective size" of the state (1) scales like $N\epsilon^2$. Note that there are several other arguments which confirm this effective size $n \approx N\epsilon^2$, e.g., an argument related to the statistical distinguishability of states $|\phi_{1,2}\rangle$ as pointed out in Ref. [21].

To summarize, we provided two different methods to assign an effective particle number to GHZ-like states of the form $|\psi\rangle \propto |\phi_1\rangle^{\otimes N} + |\phi_2\rangle^{\otimes N}$ with $|\langle\phi_1|\phi_2\rangle|^2 = 1-\epsilon^2$. The first method is based on the rate of decoherence, and we found that $|\psi\rangle$ behaves as an ideal n-party GHZ state with $n \approx N\epsilon^2$. In the second method, which is more in the spirit of quantum information, we provided lower and upper bounds for the distillation rates p_n of ideal n-party GHZ states using only local operations and classical communication. Again, we found that $\sum p_n n \approx N\epsilon^2$, i.e., the average number of particles which form an ideal n-party GHZ state, essentially scales as $n \approx N\epsilon^2$. This illustrates that not only the number of particles but also the properties of the states appearing in the microscopic description of the system determine the effective size of the corresponding GHZ-like state.

Some open questions remain. On the one hand, we have considered a particular class of GHZ-like states (1). One can have physical systems where the macroscopic superposition cannot be described by states of this form. In particular, the states appearing in the superposition may not be a tensor product of identical microscopic states, $|\phi_{1,2}\rangle^{\otimes N}$, either because they are entangled themselves, e.g., the position states of the atoms in an oscillating micromechanical cantilever or mirror, or the corresponding system cannot be decomposed in a natural way into subsystems, e.g., a superposition of two coherent states, $|\alpha\rangle + |-\alpha\rangle$. It would be interesting to study by similar means the effective size of those superpositions and maybe compare them with the ones treated here.

On the other hand, there are experiments where macroscopic superpositions have been created but there is no microscopic description of the states produced [7]. It would be interesting to find such a microscopic description and—in case the states can be written as (1) (which would be natural)—to assess an effective size of the states for these experiments.

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[20] One can use the above method to obtain also a bound for each p_n, i.e., assuming that only n-party GHZ states for some fixed n are created by the distillation protocol. One obtains $p_n \le -N\epsilon^2\log_2(\epsilon)/(2n)$ and finds that $p_n \approx 1$ if $n \le -N\epsilon^2\log_2(\epsilon)/2$, i.e., only in this case it is in principle possible to produce n-party GHZ states with a rate per copy approximately equal to 1.
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