Experimental Characterization of the Transition to Phase Synchronization of Chaotic CO_2 Laser Systems

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We investigate the transition route to phase synchronization in a chaotic laser with external modulation. Such a transition is characterized by the presence of a regime of periodic phase synchronization, in which phase slips occur with maximal coherence in the phase difference between output signal and external modulation. We provide the first experimental evidence of such a regime and demonstrate that it occurs at the crossover point between two different scaling laws of the intermittent-type behavior of phase slips.

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The study of synchronization phenomena in chaotic systems has received much attention over the past decade [1–5]. Its relevance is witnessed by the large number of studies that have demonstrated the existence of synchronized chaotic states in laboratory experiments [6], as well as in natural phenomena [7].

Synchronization of chaos refers to the fact that several chaotic units, either coupled or forced, even though keeping a general chaotic trend, correlate strongly with each other. This ranges from perfect hooking of the chaotic trajectories of two coupled systems, or complete synchronization [1], to the emergence of a functional relationship relating the chaotic outputs, or generalized synchronization [2], to a looser form of correlation consisting in the locking of the phases of the chaotic systems, or phase synchronization [3,4].

Among the above quoted behaviors, we focus on phase synchronization (PS). Here, a coupling or a forcing induce a phase locked regime, where the amplitudes remain chaotic and almost uncorrelated and the difference between the two free running (unwrapped [3]) phases $\phi_{1,2}$ is bounded, obeying the synchronization condition

$$\Delta \equiv |\phi_1 - \phi_2| < \text{const.} \quad (1)$$

PS was first observed in mutually coupled [3] or periodically forced chaotic oscillators [4] and then extensively studied in theoretical models [8] and experiments [9,10]. In particular, the occurrence of PS has been shown to play a crucial role in many physiological systems, such as human heartbeat and respiration [11], magnetoencephalography and electromyography of Parkinsonian patients [12], and electroencephalograms during visual stimulations [13].

Since PS is the weakest stage of synchronization, a relevant issue is to understand the transition route to such a behavior from unsynchronized motion. On the border of PS, the evolution of Δ is characterized by intermittent time intervals of phase synchronized motion interrupted by sudden 2π jumps in Δ, or *phase slips.* In the classical case of coupled or forced periodic oscillators, the transition to PS corresponds to a saddle-node bifurcation, and the average duration τ between successive phase slips obeys a type-I intermittency scaling law [14]

$$\tau \sim |P - P_c|^{-1/2}, \quad (2)$$

P being either the coupling strength or the frequency of the driving signal, and P_c denoting its transitional value to PS.

For chaotic systems, however, a different scenario emerges. Let us consider for the time being a forced chaotic oscillator, and let us call ν_c the value of the forcing frequency that marks the transition to PS; i.e., let us suppose that the system is phase synchronized for $\nu < \nu_c$. When considering forcing frequencies $\nu \geq \nu_c$, another transition point $\nu_t > \nu_c$ exists such that, for $\nu > \nu_t$, the scaling law for τ is the same as the classical case ($\tau \sim |\nu - \nu_t|^{-1/2}$), while for $\nu_c \leq \nu < \nu_t$, the intermittency shifts from type-I to that of superlong laminar periods described by

$$\ln\tfrac{1}{\tau} \sim -|\nu - \nu_c|^{-1/2}. \quad (3)$$

The theoretical picture of the transition to PS has been identified for a pair of coupled oscillators as a boundary crisis mediated by an unstable-unstable pair bifurcation [15], and the two above scaling behaviors have been numerically reported for coupled chaotic model systems [16].

Another important feature of the transition to PS is that it can be identified by inspecting the Lyapunov spectrum. Precisely, PS is set around the passage to a negative value of a Lyapunov exponent that was zero in the uncoupled or unforced regime [3]. More recently, it has been pointed out that the transition from no synchronization to PS is mediated by a regime, called *periodic phase synchronization* (PPS), where a local negativeness of a Lyapunov exponent induces phase jumps periodic in time, leading to a situation where the time intervals between successive phase slips (the epochs of temporary PS) are almost equal to one another [17].

In this Letter, we report experimental evidence of the entire transition route from no synchronization to PS. In particular, we provide evidence of the fact that the crossover point ν_t is exactly the value for the occurrence of PPS in the system, which then mediates the intermittent-type behavior of phase jumps.

The experimental setup [Fig. 1(a)] consists of a CO_2 laser tube, pumped by an electric discharge current of 6 mA and inserted within an optical cavity closed by a totally reflecting mirror and a partially reflecting one. The detected laser output intensity suitably amplified drives an intracavity electro-optic modulator that controls the cavity losses. Precisely, the feedback loop is realized by the voltage exiting a HgCdTe fast infrared diode detector, conveyed into an amplifier together with a bias voltage B_0, and driving the electro-optic modulating crystal. In these conditions, and in the absence of any further modulation, the output intensity consists of a train of homoclinic spikes repeating at chaotic times and interconnected by minor oscillations [10] [see Fig. 1(b)]. We demonstrated that the sequence of homoclinic spikes can be phase entrained by an external sinusoidal modulation [10]. In the present case, we add a square signal modulation in the pumping discharge whose amplitude provides a $\sim 2\%$ perturbation in the electric discharge current, and we enter a regime of PS by moving the frequency of the external modulation ν. The modulation is applied on a control unit of the generator [element 6 in Fig. 1(a)]. In the present operating conditions, the most significant source of noise is represented by the hardware noise in the feedback loop, which amounts to $\sim 0.15\%$ of the feedback signal. However, such a noise level does not affect significantly the occurrence of PS phenomena, and therefore it can be neglected for all measurements that will be reported in the following. While the phase ϕ_e of the external modulation evolves linearly in time ($\phi_e = 2\pi\nu t$), the phase ϕ_s of the chaotic signal is calculated by linear interpolation between successive spiking times

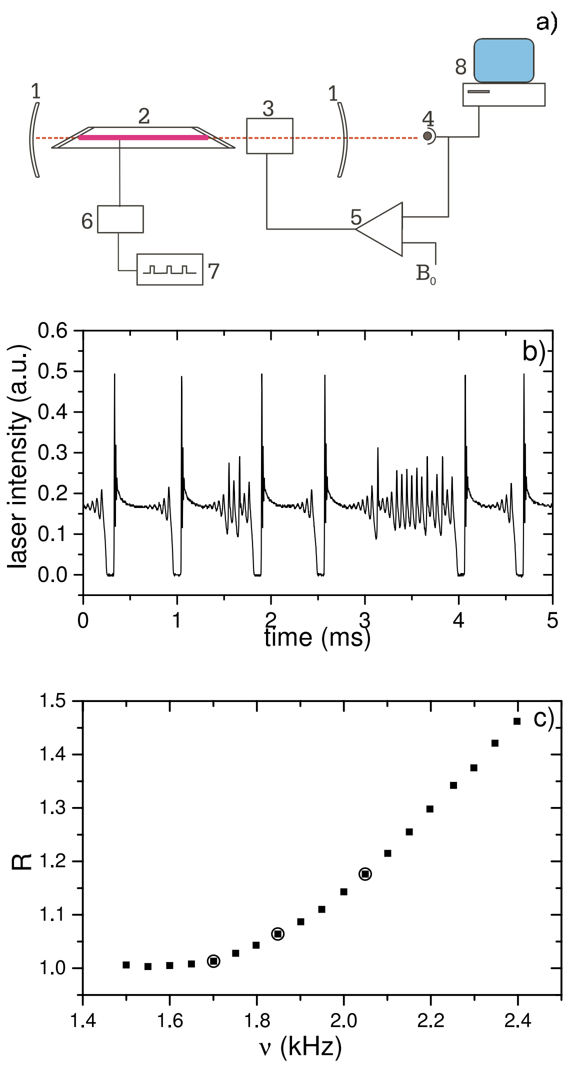

FIG. 1 (color online). (a) The experimental setup. 1: Mirrors delimiting the optical cavity; 2: CO_2 laser tube; 3: intracavity electro-optic modulator; 4: HgCdTe fast infrared diode detector; 5: amplifier; 6: generator for the pumping discharge; 7: external modulation; 8: PC based classification of phase slips; (b) laser intensity (in arbitrary units) vs time for zero amplitude modulation; (c) synchronization parameter R (see text for definition) vs ν. The three circles highlight the ν values for which phase slips are reported in Fig. 2.

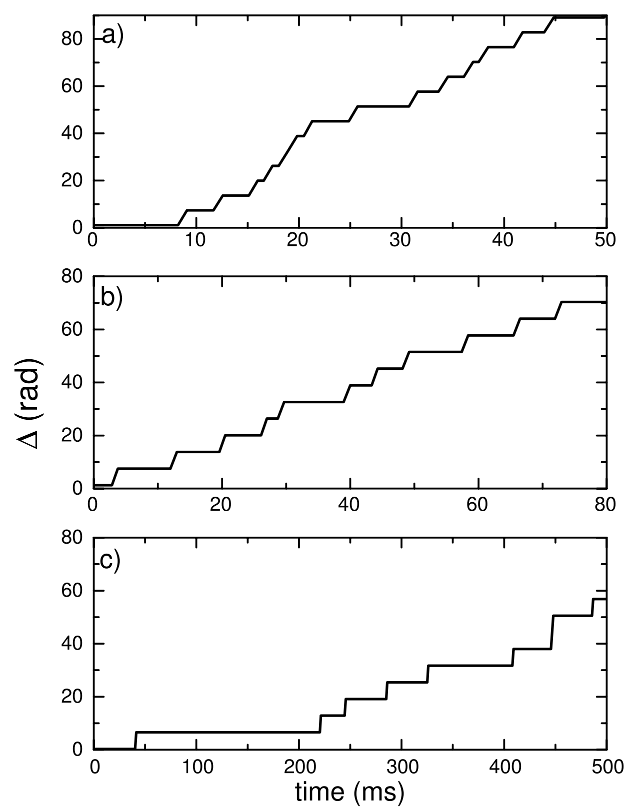

FIG. 2. Temporal evolution of $\Delta \equiv |\phi_e - \phi_s|$ for (a) $\nu = 2.05$ kHz, (b) $\nu = 1.85$ kHz, and (c) $\nu = 1.70$ kHz.

$$\phi_s = 2\pi k + 2\pi \frac{t - T_k}{T_{k+1} - T_k}, \qquad T_k \leq t < T_{k+1}, \quad (4)$$

where T_k denotes the time at which the kth spike is produced. Calling R the ratio between the number of maxima in the input modulation and the number of output spikes, Fig. 1(c) reports the route toward PS ($R = 1$) as ν approaches $\nu_c \simeq 1.62$ kHz and highlights the process of phase entrainment operated by the external modulation in the sequence of spikes.

By means of a PC based acquisition routine, we record sequences of more than 150 000 interspike intervals and study the occurrence of phase slips in the proximity of the transition point to PS.

Figure 2 reports the temporal evolution of $\Delta \equiv |\phi_e - \phi_s|$ for (a) $\nu = 2.05$ kHz, (b) $\nu = 1.85$ kHz, and (c) $\nu = 1.70$ kHz. A sequence of 2π phase slips characterizes the evolution of Δ. Their occurrence becomes rarer and rarer as ν approaches ν_c. We here provide the first experimental evidence of the existence of PPS in such a system. For this purpose, we calculate the distribution of interslip time intervals (ITI) and monitor its coherence factor

$$C \equiv \frac{\tau}{\sigma}, \quad (5)$$

as a function of ν. Here τ is the average interslip time interval, and σ the standard deviation of the ITI distribution. According to the theoretical prediction [17], one expects to have a value $\nu_{\mathrm{PPS}} > \nu_c$ where phase slips occur periodically in Δ. The above fact is reflected by a maximum in the coherence factor C close to the transition point for PS.

Figure 3 reports the behavior of $C(\nu)$. The presence of a maximum is apparent at $\nu_{\mathrm{PPS}} \simeq 1.84$ kHz, where maximal coherence is produced in the ITI distribution close to the 1:1 phase-locking regime. The further growth of C beyond $\nu \sim 2.1$ kHz is due to the approaching of a new locking regime, namely, 2:1 rather than 1:1, as illustrated in Ref. [10] for sinusoidal modulation. In this new locking regime, phase slips and their coherence need to be properly redefined in order to account for the fact that a spike in the laser output is produced for each two forcing periods. The temporal evolution of Δ at ν_{PPS} is shown in Fig. 2(b), where one can see that phase slips are almost equispaced in time.

A yet unsolved question is how the occurrence of PPS is related to the crossover between the two above discussed scaling behaviors of phase slips. The two scaling behaviors can be explained as follows. The type-I intermittency behavior of Eq. (2) describes the classical case of periodic systems. Just outside the border of phase synchronization, this power law characterizes the intermittent phase slip duration.

For chaotic systems, the PS region corresponds to the overlap of all the phase-locking regions of the unstable periodic orbits (UPO) embedded in the chaotic attractor

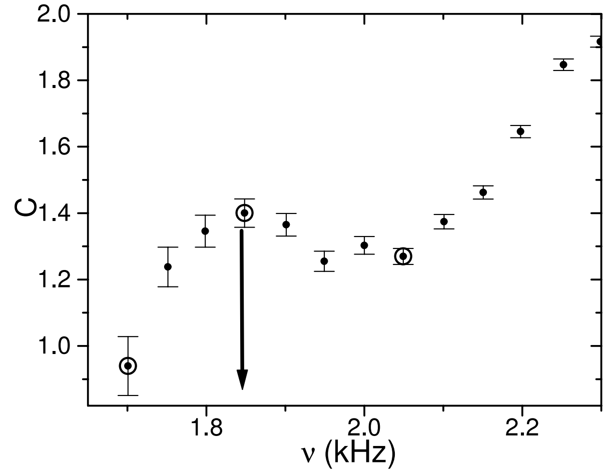

FIG. 3. Coherence factor C (see text for definition) vs external modulation frequency ν. The arrow at $\nu_{\mathrm{PPS}} \simeq 1.84$ kHz indicates the frequency value for which phase slips are maximally coherent. The circles surround the three points for which measurements of $\Delta(t)$ are reported in Fig. 2.

[18]. Each locked UPO is associated with an attractor and a repeller in the direction of the phase. The repellers are periodic orbits on the basin boundary of the attractors.

As the parameter is set close to the PS bifurcation point, the attractor and the repeller of each of a few UPOs approach, coalesce, and annihilate as a result of the saddle-node bifurcation [15]. As a result, these UPOs come out to be unlocked by the external force and phase slips may occur. Just beyond the transition point, most UPOs are still attractive, and phase slips can develop only when the chaotic trajectory comes close to an unlocked UPO. A phase slip occurs when the chaotic trajectory stays for a time τ_1 in a close vicinity of the unlocked UPO.

Now, due to ergodicity, the probability for a trajectory to visit a particular UPO for a duration τ_1 is proportional to $\exp(-\lambda\tau_1)$ (λ being the largest Lyapunov exponent). As a result, the average interslip interval (the inverse of this probability) will be given by $\tau \sim \exp(k|\nu - \nu_c|^{-1/2})$, where τ_1 has been substituted with its type-I intermittent scaling behavior, hence the superlong laminar behavior of phase slips of Eq. (3). Such a scaling behavior has been verified in the transition to PS by numerical simulation of maps [19], as well as by direct simulation of the chaotic Rössler oscillator driven by external forcing [15,18]. In other contexts, evidence of superlong transients scaling as Eq. (3) has been reported [20].

For what is said above, one should expect a type-I intermittent scaling law only for frequencies $\nu > \nu_t$, where ν_t denotes the value for which all UPOs are in the unlocked regime, so as phase slips can occur independently of the particular UPO that is visited by the chaotic trajectory. On the contrary, for $\nu_c < \nu < \nu_t$, a superlong laminar behavior occurs, since phase slips are allowed only when the chaotic trajectory stays for a

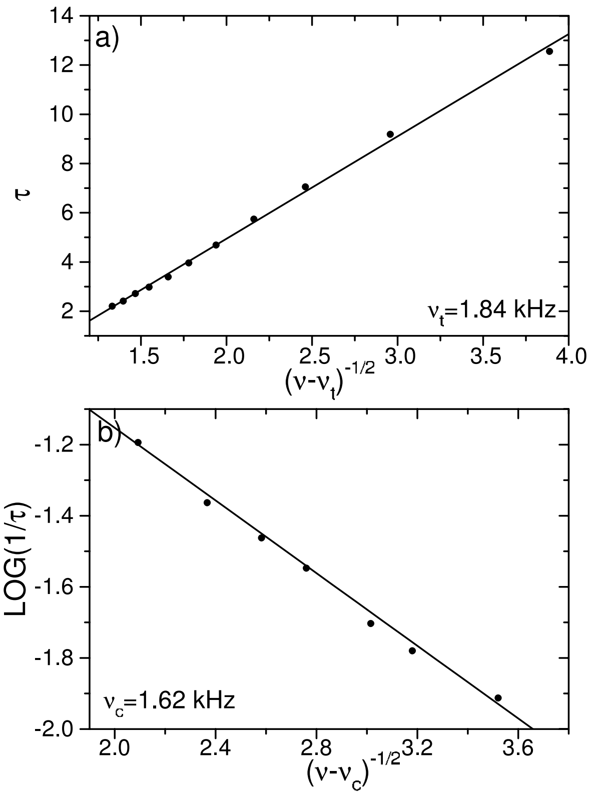

FIG. 4. Type-I intermittency scaling behavior (a) and superlong laminar scaling behavior (b) of interslip time intervals. Dots indicate the experimental measurements. Lines are the best fits $\tau = -3.4 + 4.2 \mid \nu - \nu_t \mid^{-1/2}$ [$\nu_t = 1.84$ kHz (a)] and $\log(1/\tau) = -0.13 - 0.51 \mid \nu - \nu_c \mid^{-1/2}$ [$\nu_c = 1.62$ kHz (b)]. The crossover point for the two scaling laws is located at $\nu = \nu_t = 1.84$ kHz, that corresponds exactly to the value $\nu_{\rm PPS}$ of maximal coherence (periodic phase synchronization) in the phase slip occurrence.

sufficiently long time close to those UPOs belonging to the unlocked regime.

In order to confirm the above expectations, we have performed a series of measurements at different values of ν, obtaining the results shown in Fig. 4. The best fits yield $\nu_c = 1.62$ kHz and $\nu_t = 1.84$ kHz. Besides confirming the existence of two different scaling behaviors, Fig. 4 shows that the crossover point for the two scalings coincides with $\nu_{\rm PPS}$ of Fig. 3, thus indicating that the coherence between successive phase slips mediates the transition from type-I to superlong laminar period intermittency.

In conclusion, we have given experimental confirmation of the entire transition route from no synchronized behavior to phase synchronization. In particular, we have given the first experimental evidence of periodic phase synchronization, as well as we have demonstrated its role in mediating the transition between two different scaling behaviors in the phase slip occurrence.

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