Hydrodynamic Model for a Dynamical Jammed-to-Flowing Transition in Gravity Driven Granular Media

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Granular material on an inclined plane will flow like a fluid if the angle θ the plane makes with the horizontal is large enough. We study chute flow down a plane using a hydrodynamic model previously used to describe granular Couette flow. Our model predicts a jammed-to-flowing transition as θ is increased even though it does not include solid friction, which might seem necessary to stabilize a state without flow. The transition is driven by coupling between mean and fluctuating velocity. In agreement with experiments and simulations, it predicts flow for layers with a thickness H larger than a critical value $H_{\mathrm{stop}}(\theta)$ and absence of flow for $H < H_{\mathrm{stop}}(\theta)$.

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The flow of granular materials has been the subject of intense experimental and theoretical study [1,2]. The major difficulty in describing granular flows, particularly in dense media, is that both kinetic phenomena associated with particle motion and more static phenomena such as solid friction and force chains are important. Following the pioneering work of Jenkins and Savage and of Haff [3], various kinetic-hydrodynamic theories for the coupled transport of momentum and kinetic energy have been constructed. Other authors have proposed including shear independent terms in the stress tensor to capture empirically the effects of intergrain friction [4,5]. Neither of these types of theories incorporates the effects of strongly inhomogeneous force networks in static and slowly driven dense granular materials [6,7].

To explain experimental results on sheared granular material in Couette geometry, we recently proposed a modified hydrodynamic description of granular flow [8], which included high-density effects in equations for energy and momentum transport. Though our approach does not *a priori* include enduring contacts and solid friction between grains, we were able to reproduce most of the experimental results in the Couette geometry.

Granular flow down inclined slopes is of practical importance in geophysical phenomena such as rock and snow avalanches, pyroclastic flows, etc. Experiments [9,10] and molecular simulations [11] on a layer of beads on an inclined plane as a function of layer thickness H and angle θ of the plane relative to horizontal yield a curve $H_{\mathrm{stop}}(\theta)$ separating jammed, stationary behavior at small H from flowing behavior at large H. Moreover, precise and systematic measurements of the mean-bead velocity yield a maximum velocity that scales as $H^{3/2}$.

In this Letter, we analyze chute flow down inclined planes using the hydrodynamic theory developed for the study of Couette flows [8]. Our analysis produces results in agreement with experiment and simulation in this geometry and provides further evidence that our phenomenological hydrodynamic theory provides a robust and general description of flowing granular media. In particular, we find an H_{stop} curve, or equivalently a critical angle separating stationary from flowing behavior. Normally, the existence of such a critical angle θ_c is associated with solid friction: a solid block on an inclined plane is stationary for $\theta < \theta_c$ and slides for $\theta > \theta_c$. In our analysis, the existence of θ_c is associated with a purely dynamical jammed-to-flowing transition, resulting from the balance between flow-induced "shaking" (corresponding to a viscous-like heating) and dissipation.

Geometry of the problem.—We consider the geometry of the experiments in Ref. [9]: a layer of granular material, composed of spheres of diameter d, of thickness H flows on an inclined plane making an angle θ with the horizontal. We choose a coordinate system with the x axis parallel to the plane, the z axis perpendicular to it, and origin at the bottom of the flowing layer.

In a stationary state, conservation of momentum in directions parallel and perpendicular to the plane leads to the following equations: $\partial_z \sigma_{xz} = -m\rho(z) g \sin\theta$ and $\partial_z \sigma_{zz} = m\rho(z) g \cos\theta$, where ρ is the particle number density, m is the grain mass, and σ_{zz} and σ_{xz} are, respectively, the diagonal and off-diagonal components of the stress tensor. Combining these two model-independent equations yields the simple relation $\sigma_{xz} = -\tan\theta \sigma_{zz} + C$, where C is a constant independent of z. In our hydrodynamic model, the off-diagonal component of the stress tensor is viscouslike: $\sigma_{xz} = \eta(\rho, T)\dot{\gamma}$, where $\dot{\gamma} = dV_x/dz$ is the shear rate, V_x is the mean velocity, and η is the viscosity, a function of the density and granular temperature field T. As usual the latter is defined in terms of the rms part of the velocity field $T(z) = \frac{1}{3} m \langle v^2(z) \rangle$. On the other hand, σ_{zz} is identified as minus the pressure P.

The temperature profile is determined by the balance between viscous heating, heat flow, and dissipation through inelastic collisions [8]: $\frac{\partial}{\partial z} \lambda(\rho, T) \frac{\partial}{\partial z} T + \sigma_{xz} \dot{\gamma} - \epsilon(\rho, T) T = 0$, where $\lambda(\rho, T)$ is the thermal conductivity

and $\epsilon(\rho, T)$ is the rate of kinetic energy loss [$\epsilon(\rho, T) \propto 1 - e^2$ for a granular system with a coefficient of inelasticiy e [3]]. These hydrodynamic equations are closed by the equation of state $P = f(\rho)T$. In contrast to classical thermalized fluids, there are strong temperature and density variations over the system, and the density and temperature dependence of transport coefficients plays an important role in determining flow properties. These dependences are usually obtained using kinetic theory [3] with the Enskog approximation for the collision kernel, which accounts for excluded volume but neglects any correlation between collisions (however, different forms of transport coefficients have to be considered in the astrophysical context [12]). By construction this theory yields transport coefficients that are roughly proportional to the collision frequency, i.e., to the pair-correlation function at contact, which in the high-density limit is expected to diverge as $(\rho_c - \rho)^{-1}$ close to the maximum density, ρ_c, allowed by excluded volume effects [13]. The assumption of negligible correlation between collisions should fail at high density, and in our modified description of Ref. [8], we proposed replacing the Enskog expression for the viscosity with one with a specific "anomalous" algebraic divergence, $\eta \sim (\rho_c - \rho)^{-\beta}$ [14] close to ρ_c, in full analogy with the behavior of supercooled liquids close to the glass transition [15]. This picture is supported by the experimental observation in sheared granular material of a specific scaling law relating the local shear rate $\dot{\gamma}$ to the granular temperature T [8,16].

In the Couette geometry, the density is high throughout the cell. In contrast, in the current chute geometry, small densities could in principle be reached at the top surface, and we need to include the low-density limit in our expressions for transport coefficients. Since our aim is to provide a sketch of the jammed-to-flowing transition, we do not use the full Enskog expression for transport coefficient in the small and intermediate density regions, as is done by various authors [3]. Instead, we use simple interpolating expressions that are consistent with both the high-, intermediate-, and small-density limit. Our specific choices are $\lambda(\rho, T) = [\lambda_0/(m^{1/2}d^2)](1 - \rho/\rho_c)^{-1}T^{1/2}$, $\epsilon(\rho, T) = [\epsilon_0/(m^{1/2}d)]\rho(1 - \rho/\rho_c)^{-1}T^{1/2}$, and $\eta(\rho, T) = \eta_0[m^{1/2}/d^2]\phi(\rho/\rho_c)T^{1/2}$, where m is the mass of one particle, d is its diameter, and λ_0, ϵ_0, and η_0 are dimensionless constants. The unitless function $\phi(\rho)$ is chosen so as to interpolate between the very low-density limit, where the viscosity goes to a constant, to its high-density "anomalously" diverging behavior close to ρ_c, passing through its Enskog behavior for intermediate (but already high) densities. A sensible choice is simply $\phi(\rho_r) = [\alpha(\rho_r)^{\beta-1} + \alpha_0^{\beta-1}]\alpha(\rho_r)^{-\beta}/(1 + \alpha_0^{\beta-1})$, where $\rho_r = \rho/\rho_c$, $\alpha(\rho_r) = 1 - \rho_r$, and β is the anomalous exponent discussed above. The parameter α_0 fixes the density where the crossover to the anomalous regime occurs. The equation of state is written as $P/T = \rho/(1 - \rho/\rho_c)$, which does indeed reduce to the correct limits in the low- and high-density limits [13]. We emphasize that our results are not very sensitive to the specific forms for ϕ and the equation of state. Our expressions were chosen to minimize the number of parameters in the model.

The previous equations can be made dimensionless using a reduced velocity $v = V/\sqrt{gd}$, temperature $t = T/(mgd)$, stress $\sigma = \sigma_{xz}/(m\rho_c gd)$, pressure $p = P/(m\rho_c gd)$, and distance $\tilde{z} = z/d$ (we omit the tilde over $\tilde{z}$ in the following). Using the previously defined expressions for the transport coefficients, we can now rewrite the coupled equations for stress and temperature as

$$\tfrac{\partial}{\partial z}p = -\cos\theta\tfrac{p}{p+t}\,, \tag{1a}$$

$$\partial_z[(\tfrac{p+t}{t^{1/2}})\partial_z t] + a\tfrac{\sigma^2}{\phi(\frac{p}{p+t})t^{1/2}} - bpt^{1/2} = 0, \tag{1b}$$

where $p/(p + t) = \rho/\rho_c$ is the reduced density, $a = (\rho_c d^3)^2/(\lambda_0\eta_0)$ and $b = \epsilon_0\rho_c d^3/\lambda_0$ are dimensionless constants, and σ is related to p according to $\sigma = \tan\theta p + c$, with c an as yet unspecified constant. The parameters appearing in Eq. (1) are strongly constrained by the complementary experiments performed in the Couette cell [8,16]. In this case, the velocity and temperature profiles are localized close to the moving boundary over a depth of a few particle diameters. This behavior is predicted by the hydrodynamic model with a characteristic decay length for the rms velocity given by $\delta/d = (2/b)^{1/2}$ [8]. Experimentally δ is of the order of 4–5 particle diameters, yielding $b \simeq 0.1$ (in the following we use $b = 0.111$). The exponent β was determined experimentally in Ref. [8] to be 1.75, while experiments over a larger range of data by Müth [16] yielded $\beta = 1.5$. The parameter α_0 is chosen such that the crossover to the anomalous scaling occurs roughly 10% below random close packing. In practice $\alpha_0 = 0.05$ is adequate. Eventually, a is expected to be of order unity.

The hydrodynamic equations (1) have to be supplemented by boundary conditions for the velocity and temperature fields at the bottom and top surfaces. As to the conditions on the granular temperature, we showed in our previous study of the Couette cell that a vanishing "heat flux" condition at the boundaries yielded results in agreement with the experimental ones. In the present study, we thus assume $\partial_z T = 0$ at the boundaries. In the recent simulations of Silbert *et al.* [11], the temperature profile has a vanishing derivative close to the bottom and top boundaries. The previous assumption thus amounts to slightly shifting the hydrodynamic boundaries of the system, a behavior which is encountered in classical fluids [17]. The boundary conditions for the mean velocity field are *a priori* more obvious: at the wall surface, the velocity is assumed to vanish (a condition which requires a sufficiently rough wall surface [17]), while at the top surface a stress-free boundary condition is assumed, $\sigma = 0$,

yielding $\partial_z v = 0$ at $z = H$. Since, as mentioned above, the effective hydrodynamic boundary $z = H$ is located slightly inside the material, the pressure is not expected to vanish there, and it can be written as $p_0 = z_0 \cos\theta$ (in reduced units) with z_0 a *molecular* distance of the order of the diameter (unity in reduced units). To be fully consistent with the stress-free boundary for the velocity, we thus write $\sigma = \tan\theta(p - p_0)$ (i.e., fix the constant $c = -\tan\theta p_0$). As emphasized above, this is a molecular effect that disappears in the large-H limit. Moreover, this choice does not affect the existence of the jammed-to-flowing transition to be discussed now.

We solved the system of equations (1) using a standard Runge-Kutta algorithm and a shootinglike procedure: for a given temperature $T(H)$ at the top boundary, the derivative of T at the bottom, $\partial_z T(0)$ is computed. The solutions we seek thus correspond to zeros in the $\partial_z T(0)$ versus $T(H)$ curve. A transition between two kinds of behavior is then found: for a set $\{H, \theta\}$ such that H is below a critical $H_{\text{stop}}(\theta)$ curve, only the $T = 0$ solution exists. This corresponds to a "jammed" state, which stays at rest. Above this critical value, nonzero solutions for the temperature, which correspond to a flowing regime do exist. In Fig. 1, we plot the boundary between the two regimes, denoted $H_{\text{stop}}(\theta)$ as in Ref. [9]. This curve is in qualitative agreement with the experimental results of Ref. [9] and simulations of Ref. [11].

This shows that the hydrodynamic model is able to generate a finite critical angle below which no flow occurs. Within the model, the origin of this behavior is simply the balance between the "viscous heating," which generates fluctuations, and dissipation, which tends to inhibit the flow. The transition is thus purely dynamical: a jammed state occurs for low angles/thicknesses because fluctuations are insufficient to allow flow.

Above $H_{\text{stop}}(\theta)$, more than one solution can be found for some sets of parameters $\{H, \theta\}$. However, a linear stability analysis of the hydrodynamic equation for the temperature shows that only the nonzero solution with the largest value of $T(H)$ is dynamically stable. On the other hand, the $T = 0$ solution is always linearly stable but becomes unstable against larger "kicks" in the temperature. It is important to note that there is a solution to the hydrodynamic equations for any set of H and θ above the boundary $H_{\text{stop}}(\theta)$, as is found experimentally. This is in contrast to other approaches where for a given θ a *single* flowing thickness H is selected [4]. In the flowing regime, we obtain temperature and density profiles, which are plotted in Fig. 2 for $H = 40$.

Both results for the density and temperature are in qualitative agreement with the simulations of Ref. [11]. In particular the density profile is almost constant in the middle of the sample, with a large drop at the top surface, as found numerically. In the simulations, however, the plateau in the density is flatter than ours. Since $\rho(z)/\rho_c$ is always well less than 1 the thermal pressure of our model has no trouble supporting the weight of grains. Pressure from enduring contacts does, however, appear to be important in simulations [11,18] and experiments.

Velocity profiles are obtained by integrating the shear rate $\dot{\gamma} = dv_x/dz$, which in our model is linearly related to the nondiagonal stress $\sigma_{xz}(z) = \eta[\rho(z), T(z)]\dot{\gamma}(z)$. We find that the z dependence of the profiles is in very good agreement with the predictions of a simple Bagnold scaling $\sigma = A_{\text{Bag}}\dot{\gamma}^2$, $v(z)/v_{\max} = 1 - (1 - z/H)^{3/2}$. Within the Bagnold assumption, $v_{\max} = (2/3)A_{\text{Bag}}\sqrt{\sin\theta}H^{3/2}$. This agreement simply indicates that *once the system flows,* Bagnold-like scaling is a good approximation to the flow properties, even though *it cannot predict a jammed-to-flowing transition.* The same agreement with Bagnold scaling was observed in the simulations of Ref. [11]. The $H^{3/2}$ dependence of the maximum velocity $v_{\max}$ is also recovered in our model in agreement with experiments and simulations. Our results are also consistent with a $\sqrt{\sin\theta}$ dependence of $v_{\max}$ with $A_{\text{Bag}} \approx 0.2$. This is in fair agreement with the simulations of Silbert

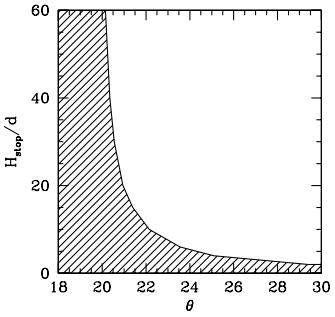

FIG. 1. Phase diagram for the jammed-to-flowing transition in a gravity driven granular material as obtained from the resolution of the hydrodynamic equations (1). The parameters used are $a = 1.3$, $b = 0.111$, $\alpha_0 = 0.05$, $\beta = 1.5$, and $z_0 = 0.5$.

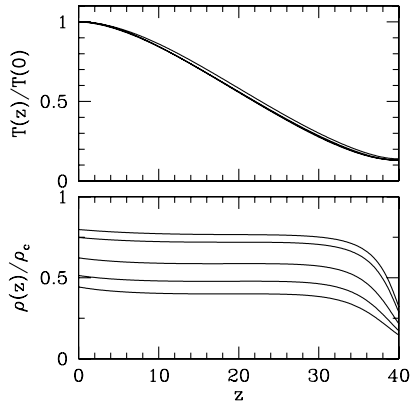

FIG. 2. Top: Normalized temperature profiles for $H = 40$, $\theta = 20.55, 20.90, 21.50, 22.50$. Bottom: Normalized density profiles for $H = 40$, $\theta = 20.55, 20.90, 22.50, 24.50, 26.50$ from top to bottom. ($a = 1.3$, $b = 0.111$, $\alpha_0 = 0.05$, $\beta = 1.5$, and $z_0 = 0.5$.)

et al. [11], who find a slight dependence of A_{Bag} on θ. On the other hand, the experiments of Pouliquen exhibit a stronger dependence on θ, with $v_{\max}$ scaling like $[H_{\mathrm{stop}}(\theta)]^{-1}$.

Discussion.—Reference [8] introduced a modified kinetic-hydrodynamic theory for granular flow that predicted temperature and velocity profiles in dense systems in Couette geometry in agreement with experiments. In this Letter, we calculate profiles for flow down an inclined plane using a generalization of this theory to include gravity and low- as well as intermediate- and high-density regimes. Our results agree with those obtained in experiments [9] and simulations [11]. The hydrodynamic theory of granular flow thus provides a good description of flow in various qualitatively different geometries. In the case of flow down a plane, it is able, contrary to intuition, to predict a jammed-to-flowing transition and a critical angle below which no flow takes place.

This success is astonishing at first since our model is based on a kinetic-hydrodynamic theory for inelastic systems assuming binary collisions and neglecting enduring contacts [3]. Thus static solid friction effects between grains are not *a priori* included in the description. However, as stated above, the key ingredient to explain the jammed-to-flowing transition within our description is the balance between flow-induced "*shaking*" and *dissipation*: the latter originates in the internal degrees of freedom of the grains, and both inelasticity of collisions and surface friction are macroscopic expressions (at the level of a grain) of this effect. Our proposed mechanism for the jammed-to-flowing transition is, therefore, expected to remain valid when solid friction is taken explicitly into account since it acts like an inelastic loss term in the time evolution of fluctuations. This assertion is supported by the excellent qualitative agreement between the present description and simulation results for systems with surface friction in both the Couette [19] and gravity driven [11] cases. The general features of the present description should, therefore, remain when solid friction is included, although some details (such as the functional form of the transport coefficients) might need appropriate corrections [see, e.g., Ref. [2](c) for an attempt to go beyond kinetic approaches]. In particular, it seems desirable to extend the hydrodynamic description to include enduring contacts, which can be included in transport equations (see, e.g., [20]). Their inclusion will lead to a modification of the $T^{1/2}$ scaling of transport coefficients, which follows from the assumption of binary collisions. More generally, one might intuitively expect that the coupling between fluctuations and mean flow, which underlies the present description, should be a generic feature of rheology of glassy systems: in order to flow, the system must create its own fluctuations, which in turn couple to flow. It is interesting to note that various descriptions based on similar ideas have recently emerged in attempts to provide a phenomenological description of the rheology of complex systems such as gels, paste [21], and foam [22], where similar behaviors (in particular, heterogeneous flow properties) have been observed.

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