

Gap Bifurcations in Nonlinear Dynamical Systems

F. B. Rizzato^{1,*} and R. Pakter^{1,2,3}

¹*Instituto de Física, Universidade Federal do Rio Grande do Sul, P.O.Box 15051, 91501-970, Porto Alegre, RS, Brasil*

²*Departamento de Matemática, Universidade Luterana do Brasil, Av. Miguel Tostes 101, 92420, Canoas, RS, Brasil*

³*Universidade Estadual do Rio Grande do Sul, Rua Inconfidentes 395, 93340-000, Novo Hamburgo, RS, Brasil*

(Received 29 March 2002; published 10 October 2002)

We investigate the dynamics generated by a type of equation which is common to a variety of physical systems where the undesirable effects of a number of self-consistent nonlinear forces are balanced by an externally imposed controlling harmonic force. We show that the equation presents a new sequence of bifurcations where periodic orbits are created and destroyed in such a nonsimultaneous way that may leave the appropriate phase-space occasionally empty of fundamental harmonic orbits and confined trajectories. A generic analytical model is developed and compared with a concrete physical example.

DOI: 10.1103/PhysRevLett.89.184102

PACS numbers: 05.45.-a, 41.85.Ja

There is a type of equation which is very common to a variety of physical systems. It essentially describes the combined action of a number of nonlinear force terms along with a time dependent harmonic force on the dynamics of an s -dependent radial-like physical quantity $r = r(s)$. While the nonlinear forces encompass self-consistently generated repulsive effects which tend to disorder and eventually destroy the system, the harmonic force models limiting effects externally imposed in order to control the system. The equation reads

$$\frac{d^2 r}{ds^2} + k^2(s)r = F(r). \quad (1)$$

$k(s)$ is a periodic function satisfying $k(s+1) = k(s)$, with time average $\overline{k^2(s)} = k_0^2$; k_0 is constant and the period has been normalized to the unity. $F(r)$ contains all nonlinear forces acting on the system—including inertial forces such as angular momenta—and must be such that enforces the radial condition $r \geq 0$; we simply take F as a hard-core-type force satisfying $F(r \rightarrow 0) \rightarrow +\infty$. We also require that $F(r \rightarrow \infty) \rightarrow 0$ since this condition enables the theory to describe a large number of systems with decaying fields as in the cases of Coulombic or gravitational interaction. This condition also guarantees that one has at least one equilibrium when $k(s)$ is constant, $k(s) = k_0$, which we suppose to be the only one available. We also observe the twist condition [1] and suppose that the frequency curve measured from the equilibrium—again when $k(s) = k_0$ —is a monotonic function of the appropriate action, which in practice results from all the previous hypothesis.

The purpose of the present Letter is to introduce and analyze bifurcations of Eq. (1) along the lines to be explained shortly. We first mention that one instance where Eq. (1) can be found is in the case of a Paul trap, a device designed to confine ions [2–4]. Here s is simply the time, $r \equiv \sqrt{x^2 + y^2}$ is the magnitude of the transverse (with respect to a certain z axis) radius of the ion orbital

trajectory, and F incorporates the ion-ion electrostatic interaction within the trap in the form $F \sim 1/r^2$ [3]. Another example comes from the area of physics of beams, where an equation such as Eq. (1) is obtained for the axially symmetric envelope of particle beams confined by external solenoidal magnetic fields [5,6]. s in this case represents the z coordinate along which the beam is transported, $k(s)$ represents the action of a confining external magnetic field, and F incorporates thermal and self-consistent effects of the beam: $F = 1/r^3 + K/r$ where K is known as the beam perveance associated with self-fields and where the cubic term describes thermal emittance. Self-fields and emittance are defocusing factors that must be compensated with the use of focusing magnetic fields.

Now, it is a matter of interest to determine orbits displaying the same periodicity of the external field described by the parameter $k(s)$; we shall refer to these orbits as the fundamental harmonic orbits (FH orbits for short). In electrostatic confining devices FH orbits are known as π orbits, and in physics of beams they are known as matched solutions. We dedicate this Letter to the analysis of possible bifurcations that either destroy or create FH orbits. In particular, we shall see that depending on the relevant parameters, gaps may be formed as one changes the confining fields, within which no FH orbits are present. This fact had already been numerically recognized in a previous work [6]. Here we generalize the concept and provide an analytical framework for the whole nonlinear process which we call *gap bifurcation*, for short, when gaps are indeed created. We shall see how a gap bifurcation is related to creation and extinction of periodic orbits.

Equation (1) can be canonically derived from the Hamiltonian principle

$$\begin{aligned} r' &= \partial H / \partial p = p, & p' &= -\partial H / \partial r, \\ H(p, r) &= p^2/2 + k(s)^2 r^2/2 + U(r), \end{aligned} \quad (2)$$

where $F(r) = -dU/dr$ and where primes denote derivatives with respect to s from now on. When $k(s) = k_0$ the system becomes integrable, but even in this case for a generic $U(r)$ it is very unlikely that we can find closed analytical solutions for the problem. Since the gap effect we intend to explore here is independent of the minute details of U , let us resort to an approximative approach. First we note that in the static case $k(s) = k_0$, a central fixed point r_0 can be obtained as the solution of $r'' = 0 \Rightarrow -k_0^2 r_0 + F(r_0) = 0$. Orbits oscillate around r_0 bounded by minimum and maximum elongations at $r' \equiv p = 0$; r_{\min} and r_{\max} , respectively. Each orbit can be indexed by the corresponding action J alluded to earlier, such that increasingly larger values of r_{\max} or smaller values of r_{\min} are both associated to increasingly larger values of J . The oscillatory frequency for orbits near the fixed point can be found in the form $\omega = \omega_0 = (k_0^2 - dF/dr|_{r=r_0})^{1/2}$, and the frequency of oscillations with maximum elongation satisfying $r_{\max} \rightarrow \infty$ or $r_{\min} \rightarrow 0$ reads $\omega = \omega_\infty = 2k_0$. ω_∞ has this particular form if one considers that while for large r 's F has no effect on the orbit which therefore moves harmonically, for small r 's the hard-core portion of F builds up a reflecting intransposable barrier—hence the factor of “2” in ω_∞ . The idea here is to see if periodic orbits with the same periodicity of the driver can be created. Periodicity can be better analyzed with help of action-angle coordinates. Using our hypothesis on the monotonicity of the frequency curve $\omega = \omega(J)$, one can interpolate a convenient formula between the central and asymptotic frequencies ω_0 and ω_∞ as follows:

$$\omega(J) = \omega_0 + \Delta \varepsilon J (1 + \varepsilon J)^{-1}, \quad (3)$$

where $\Delta \equiv \omega_\infty - \omega_0$, and $\varepsilon > 0$, yet undetermined, accounts for the rate with which the frequency changes with the action J . Then the unperturbed Hamiltonian $H_0(J)$ can be obtained as

$$H_0(J) = \int_{J'=0}^J \omega(J') dJ' = (k_0 + \Delta)J - \varepsilon^{-1} \Delta \ln(1 + \varepsilon J). \quad (4)$$

The factor ε can be obtained as soon as one has concrete information on the form of $H(p, r, k = k_0)$ expressed in terms of the corresponding action variable for small values of this action and $k = k_0$, and this can be done perturbatively. Then it is a simpler matter to evaluate $\varepsilon = (1/\Delta) \partial^2 H(k = k_0) / \partial J^2|_{J=0}$. We shall analyze one concrete case shortly but for now let us switch on the varying contribution of the external field with $k^2(s) - k_0^2 \equiv \delta k_0^2 \cos 2\pi s$, and analyze the existence and bifurcations of FH orbits. δ is a constant amplitude factor satisfying $0 \leq \delta \leq 1$ and we choose a periodic harmonic function for simplicity—results arising from other periodic functions are qualitatively similar. We write

$$H(p, r) = H_u + H_p \quad (5)$$

with $H_u \equiv p^2/2 + k_0^2 r^2 + U$ and $H_p \equiv k_0^2 \delta \cos(2\pi s) r^2$.

H_u is replaced with H_0 of Eq. (4) and r is written in the form $r = r_0 + w$. Now we need an expression for w in terms of action-angle variables, $w = w(J, \theta)$. If w is small, this poses no problem and we can simply write

$$w = \sqrt{2J/\omega_0} \cos \theta \quad \text{and} \quad p = -\sqrt{2J/\omega_0} \sin \theta \quad (6)$$

because orbits are nearly circular near the equilibrium at $r = r_0$, $p = 0$. When w is large, on the other hand, we have seen that the motion is similar to that of an oscillator blocked at the origin. The oscillator moves freely when $w > 0$, but is reflected at the origin. Therefore we attempt to represent its orbit in the form

$$w = A(J) |\cos(\theta/2)|; \quad (7)$$

as θ moves from $-\pi$ to $+\pi$ crossing $\theta = 0$, w goes from $w = 0$ at $\theta = -\pi$, crosses its maximum, $A(J)$, at $\theta = 0$, and returns to $w = 0$ at $\theta = +\pi$ —the cycle is repeated then. $A(J)$ is determined when one realizes that for large values of J and at maximal radius r_{\max} , where $p = 0$ and $\theta = 0$, $H_u = \omega_\infty J \approx k_0^2 A(J)^2/2$ which implies $A(J) = \sqrt{8J/\omega_\infty}$. This is different from the purely harmonic case where one would have a multiplying factor of “2” instead of “8” under the square root in the expression for $A(J)$. We now proceed to construct the resonant Hamiltonian for the FH orbits. When J is small one uses expressions (6) in H_p and collects only those harmonic terms with argument $\theta - 2\pi s$:

$$H_p(J \rightarrow 0) \approx (1/2) \delta k_0^2 r_0 \sqrt{2J/\omega_0} \cos(\theta - 2\pi s). \quad (8)$$

When J is large, we select the more significant r^2 term and write

$$\begin{aligned} H_p(J \rightarrow \infty) &\approx \frac{1}{2} \delta k_0^2 \left(\frac{8J}{\omega_\infty} \right) \cos^2(\theta/2) \cos(2\pi s) \\ &= \frac{1}{4} \delta k_0^2 \left(\frac{8J}{\omega_\infty} \right) (1 + \cos \theta) \cos(2\pi s) \\ &\rightarrow \delta k_0^2 \left(\frac{J}{\omega_\infty} \right) \cos(\theta - 2\pi s), \end{aligned} \quad (9)$$

where we dropped off-resonant terms in the last step again. Now we do not know the crossover details from expression (8) to (9), although the issue lacks importance here. The reason is that for small J the \sqrt{J} term automatically dominates, while for J large it is the linear J term that automatically prevails. In any case we use the additional canonical transformation $\theta - 2\pi s \rightarrow \theta$ and $H(J, \theta) - 2\pi J \rightarrow h_{\text{res}}$ to write a final form

$$h_{\text{res}} = H_0(J) - 2\pi J + \delta k_0^2 \left[\frac{r_0}{2} \sqrt{\frac{2J}{\omega_0}} + f(J) \frac{J}{\omega_\infty} \right] \cos \theta, \quad (10)$$

where we introduce a crossover factor f modeled by $f(J) = J^\xi / (1 + J^\xi)$, with ξ to be chosen in order to refine the already nice agreement with simulations of Eq. (1). FH orbits are resonances at the fundamental harmonic

which manifest themselves as fixed points of the dynamics entailed by h_{res} . Fixed points are defined in the form $\theta' = J' = 0$, and from Eq. (10) along with the proper canonical equations this demands:

$$\theta_{\text{fixed}}^{(+)} = 0, \theta_{\text{fixed}}^{(-)} = \pi, g_{\pm} \equiv \frac{d}{dJ} h_{\text{res}}[J, \theta_{\text{fixed}}^{(\pm)}] = 0. \quad (11)$$

Set (11) along with the resonant form (10) is basically where we wish to arrive. As far as overlap with other resonances not being significant, the set provides all the analytical information on bifurcations occurring with harmonic resonances. The task we have ahead is to investigate its content and compare it with simulations based on Eq. (1). As stated earlier, set (11) tells us that fixed points can be seen as zeros of functions g_{\pm} . These functions are parametrized by several control factors and their shapes depend on those factors. Perhaps the most adequate procedure here is thus to draw the functions g_{\pm} against the running argument J and examine its behavior as we vary these control parameters. We note initially that the functions have some general properties. When $\delta \neq 0$ they satisfy $g_{+}(J) > g_{-}(J)$, $g_{\pm}(J \rightarrow 0) \rightarrow \pm\infty$, and $g_{\pm}(J \rightarrow \infty) \sim 2k_0 - 2\pi \pm \delta k_0/2$ which indicates an asymptotic separation of the form $g_{+} - g_{-} \rightarrow \delta k_0 \equiv \Delta_g$ as $J \rightarrow \infty$. It is true that g_{+} or g_{-} have curvatures, but if Δ_g is sufficiently large, the minimum of g_{+} will lie above the maximum of g_{-} . Under these circumstances a gap will be formed within which no fixed point for the harmonic resonance can be found and this is the case we refer to as gap bifurcation. All this is of relevance if one is interested in finding FH orbits. To have an idea of the necessary conditions for gap formation, one can think as follows. Consider $\Delta > 0$ —the reasoning is similar if $\Delta < 0$. g_{-} then monotonically increases with J but curve g_{+} does not when $\delta \neq 0$. As one approaches $J = 0$ from large (and positive) values of J , the unperturbed frequency $\omega(J)$ provides a negative contribution to function g_{+} while the δ contribution is positive as explained earlier. Therefore a minimum may be formed if the δ/\sqrt{J} term is still small when the curvature due to $\omega(J)$ is noticeable, i.e., at $J \sim 1/\varepsilon$. Then one concludes that the minimum is present when

$$\frac{\varepsilon \Delta / \varepsilon}{1 + \varepsilon(1/\varepsilon)} \gtrsim \frac{\delta k_0^2 r_0}{\sqrt{\omega_0 / \varepsilon}} \Rightarrow \Delta \gtrsim \Delta_c \equiv \delta k_0^2 r_0 \sqrt{\frac{\varepsilon}{\omega_0}}; \quad (12)$$

there is no minimum if $\Delta \ll \Delta_c$. The gap is always present in the absence of a minimum since in this case g_{+} decreases monotonically as J increases. When the minimum is present and well pronounced, on the other hand, the gap exists if $\Delta_g > \Delta$ since in this case the downwards curvature of g_{+} is smaller than the asymptotic separation between g_{+} and g_{-} .

From the discussion above, it is clear that the presence of gaps is a direct result of the asymmetry provided by the ‘‘centrifugal’’ or hard-core portion of $F(r)$. This hard-core contribution shifts the equilibrium from the origin to

some point along the $r' = 0$ axis and blocks the dynamics at $r = 0$ such that a representation like the one of Eq. (7) becomes true. It is this kind of representation, combined with the dominant r^2 term, that generates the Δ_g separation and the possibility of the gap bifurcation. In that sense, Hamiltonian (2) is in fact structurally unstable because any modification on the r dependence of the term multiplied by $k(s)^2$ will affect the asymptotic behavior of the theory and the onset of the gaps. However, if one is interested in limited ranges of variation for r , as it is typically the case in practical systems, the quadratic approximation should be sufficiently accurate.

Up to the present point no mention was made of any particular system so the results are general. But now we finally test the theory against simulations of the original Eq. (1). In the simulations we launch several initial conditions at $p = 0$, integrate forward these conditions until their next to first return to $p = 0$, and compute the associated s interval S . From the integration we obtain the approximate differential frequency $\nu = 2\pi/S - 2\pi$ —which are equivalent to curves g_{\pm} —and the approximate action $J = (1/2\pi) \oint p dr = (1/2\pi) \int_0^S p^2 ds$. The results are then compared with those obtained from the set (11). We superpose the computed results with the estimates in the same figure. To perform the comparisons we analyze a concrete case of beam transport. We choose two sets of values for the control parameters K , k_0 , and δ in order to cover all the relevant possibilities. It will be shown in a future publication how ε is obtained as outlined just after Eq. (4); $\xi = 0.5$. Let us start with Fig. 1(a) where $K = 0.5$, $k_0 = 182.73^\circ$, and $\delta = 0.03$. In this case $\Delta/\Delta_c \approx 3.43 \sim O(1)$ and $\Delta - \Delta_g \approx +0.13 > 0$, so we expect to see a nonmonotonic g_{+} function and the absence of the gap. This is what Fig. 1(a) indeed reveals. Both analytical calculations and simulations indicate the absence of gap and the presence of a prominent downwards curvature of g_{+} . The two branches of ν correspond to initial conditions launched on the right (upper branch) and left (lower branch) sides of r_0 . We state here that the entire set of curves basically shift vertically upwards without any appreciable changes of its shape as k_0 rises within narrow ranges. Figure 1 allows one to conclude that two of the three fixed points present in the system are either born or vanish through tangent bifurcations [7] and that the tangencies do occur while the third point is already present in the system ($s_{1,2}$ denotes the two stable points and u the unstable one). We then look at a complementary situation where tangencies and gaps are simultaneously present; this is done with the help of Fig. 1(b) where $K = 3.0$, $k_0 = 180^\circ$, and $\delta = 1.0$ with $\Delta/\Delta_c \approx 0.5 \sim O(1)$ and $\Delta - \Delta_g \approx -2.11 < 0$. Here the separation Δ_g is chosen large enough that the curvature of g_{+} is not sufficient to close the gap. As we increase k_0 both fixed points associated with g_{+} undergo an inverse tangency and no fixed point survives until the one associated with g_{-} makes it appearance later on. In Fig. 2 we briefly compare the numerically obtained differential frequencies ν , but now

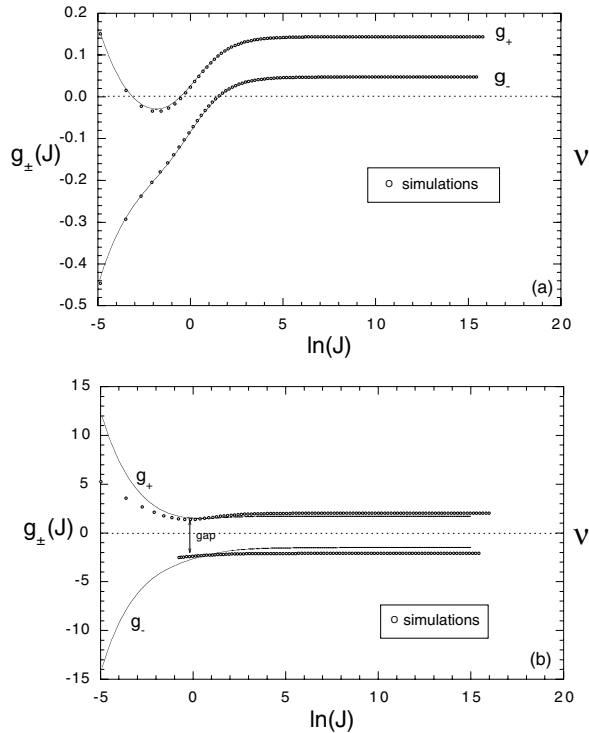


FIG. 1. g_{\pm} and ν versus $\ln(J)$. $K = 0.5$, $\delta = 0.03$, and $k_0 = 182.73^\circ$ in (a), and $K = 3.0$, $\delta = 1.0$, and $k_0 = 180^\circ$ in (b).

represented as functions of the coordinate r , with Poincaré plots on the phase-space p vs r . All the parameters are the same as the ones used in Fig. 1(a); in Fig. 2(a) we plot the simulated differential curves ν vs r and in Fig. 2(b) we display the corresponding Poincaré plot. Inspection shows how accurately the position of fixed points have been determined with the analytical model. Further plots corresponding to parameters of Fig. 1(b) show only orbits escaping to $+\infty$, which clearly corresponds to the absence of entrapping fixed points within the gap, in accordance to the model theory. Figure 1 provides a graphic way to visualize the bifurcation: the distance between g_+ and g_- is essentially set by Δ_g and K , and the birth, existence, and extinction of FH orbits result from k_0 variations as an ordered sequence of local saddle-node bifurcations and saddle nodes at $r \rightarrow 0$ and $r \rightarrow \infty$, involving orbits of different origins [bifurcation diagrams can be obtained from Fig. 1]. Gaps are formed when orbit extinction occurs before new orbits are created and this results in a global process since then all closed orbits present for sufficiently small and large values of k_0 undergo a flow rearrangement and open up within the gap. Given its global character the bifurcation is not only a mathematical technicality, being of direct physical relevance: gaps must be avoided in physics of beam, for instance, since they establish a parametric region within which the beam radius would grow without bounds as discussed in Ref. [6].

To conclude, we have analyzed the dynamics dictated by a class of equations of the form given by Eq. (1), which

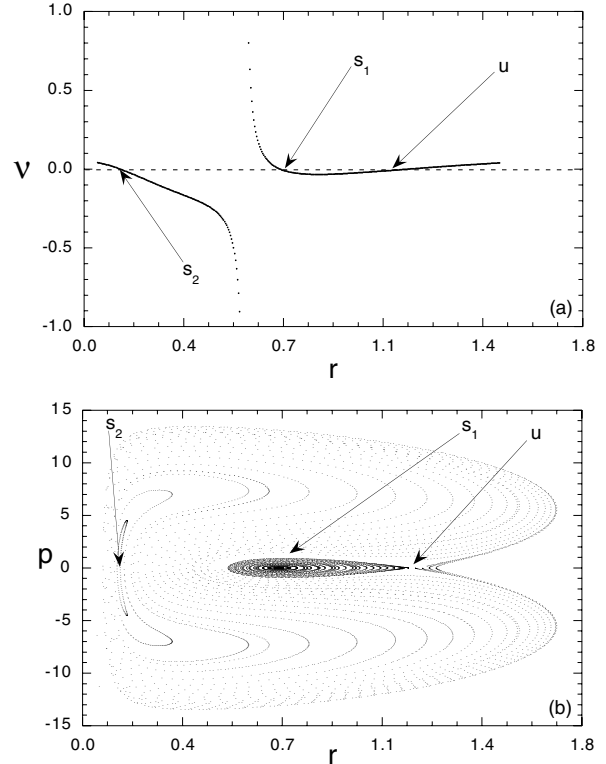


FIG. 2. A comparison of ν curves with the phase plot using the same parameters as Fig. 1(a).

is very common to a variety of physical systems. It was shown that under some very general restrictions, gaps may exist as we vary the relevant control parameters within which no solution displaying the same periodicity of the external drive can be possible. We call this non-linear mechanism gap bifurcation. Knowledge of the gap is of relevance in characterizing the stability of the system, and is crucial at least in the case of confining devices such as beam transport systems and Paul traps, whose relevant dynamics is described by radial-like equations such as Eq. (1).

This work was partially supported by CNPq, Brasil.

*Email address: rizzato@if.ufrgs.br

- [1] J. D. Meiss, *Rev. Mod. Phys.* **64**, 795 (1992).
- [2] W. Paul, *Rev. Mod. Phys.* **62**, 531 (1990).
- [3] M. Košťun, W. W. Smith, and J. Javanainen, *Phys. Rev. A* **57**, 2895 (1998).
- [4] J. Hoffnagle and R. G. Brewer, *Phys. Rev. Lett.* **71**, 1828 (1993).
- [5] C. Chen and R. C. Davidson, *Phys. Rev. Lett.*, **72**, 2195 (1994); C. Chen, R. Pakter, and R. C. Davidson, *Phys. Rev. Lett.*, **79**, 225 (1997).
- [6] R. Pakter and F. B. Rizzato, *Phys. Rev. Lett.* **87**, 044801 (2001).
- [7] A. J. Lichtenberg and M. A. Leiberman, *Regular and Stochastic Motion* (Springer-Verlag, New York, 1992).