Approximate Bogomol'nyi-Prasad-Sommerfield States

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We consider dimensionally reduced three-dimensional supersymmetric Yang-Mills–Chern-Simons theory. Although the $\mathcal{N} = 1$ supersymmetry of this theory does not allow local massive Bogomol'nyi-Prasad-Sommerfield (BPS) states, we find approximate BPS states which have nonzero masses that are almost independent of the Yang-Mills coupling constant and which are a reflection of the massless BPS states of the underlying $\mathcal{N} = 1$ super–Yang-Mills theory. The masses of these states at large Yang-Mills coupling are exactly at the n-particle continuum thresholds. This leads to a relation between their masses at zero and large Yang-Mills coupling.

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Bogomol'nyi-Prasad-Sommerfield (BPS) states play an important role in modern quantum field theory for a variety of reasons. In particular, their property of having masses which are independent of the coupling is a very useful tool, because it allows the evaluation of the spectrum in the nonperturbative region where it is otherwise difficult if not impossible to solve the theory. BPS saturated states arise because the supersymmetry protects the masses of states which are destroyed by a linear combination of the supercharges while forcing these masses to be equal to the central charge in appropriate units. It is well known that one must have at least $\mathcal{N} = 2$ supersymmetry to have a nonzero central charge and therefore massive BPS saturated states. The one exception to this is solitons, which are unrelated to the solutions we are considering [1]. Many of the more interesting field theories, at least from the phenomenological point of view, only have $\mathcal{N} = 1$ supersymmetry, and this magical property of BPS states is of no help in calculating the massive spectrum in the nonperturbative regime.

We will show by explicit demonstration that it is possible to find *approximate* BPS states in theories with $\mathcal{N} = 1$ supersymmetry. These are states with masses that are to a very good approximation independent of the coupling but are not true BPS states. We will show that the origin of these states is the presence of massless BPS states in the closely related $\mathcal{N} = 1$ supersymmetric Yang-Mills (SYM) theories. These theories have BPS saturated states that are massless and are destroyed by one of the supercharges.

For this demonstration, we consider dimensionally reduced SYM–Chern-Simons theory. This theory has the advantage that the partons are given a bare mass without breaking the supersymmetry, which makes the theory particularly suitable for numerical studies. However, such theories are also interesting in their own right. A Chern-Simons (CS) theory can be used to study many interesting phenomena, such as [2] the quantum Hall effect, Landau levels, nontrivial topological structures, vortices, and anyons. According to Witten [3], it is possible that string field theory is essentially a noncommutative CS theory. This idea led to a conjecture by Susskind [4] that relates string theory to the fractional quantum Hall effect. The SYM-CS theories are particularly remarkable. As is well known, there is a finite anomaly that shifts the CS coupling [5]. Moreover, Witten [6] has conjectured that this theory spontaneously breaks supersymmetry for some values of the CS coupling.

In this Letter, we briefly discuss SYM-CS theory in 2 + 1 dimensions but then dimensionally reduce it to two dimensions by requiring all of the fields to be independent of the transverse coordinate. This reduction eliminates many of the most interesting aspects of CS theory, including the quantization of the CS coupling, but does preserve the fact that the CS term simulates a mass for the theory. The effective mass leads to QCD-like properties for the theory, which are discussed in [7].

The numerical method we use to solve the SYM-CS theory is supersymmetric discrete light-cone quantization (SDLCQ). This method can be used to solve any theory with enough supersymmetry to be finite. By the use of ordinary discrete light-cone quantization (DLCQ) [8,9] we can construct a finite-dimensional representation of the superalgebra [10]. From this representation of the superalgebra, we construct a finite-dimensional Hamiltonian which we diagonalize numerically. Unlike direct discretization of the Hamiltonian, this construction automatically preserves supersymmetry exactly. We repeat the construction for larger and larger representations and extrapolate the solution to the continuum. We have already used this method to solve (1 + 1) and (2 + 1)-dimensional SYM theories [11–15] (for additional references to this work, see [7]), as well as the dimensionally reduced SYM-CS theory [7].

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In constructing the discrete approximation, we drop the longitudinal zero-momentum mode. For some discussion of dynamical and constrained zero modes, see the review [9] and previous work [11,12]. Inclusion of these modes would be ideal, but the techniques required to include them in a numerical calculation have proven to be difficult to develop, particularly because of nonlinearities. For DLCQ calculations that can be compared with exact solutions, the exclusion of zero modes does not affect the massive spectrum [9]. In scalar theories, it has been known for some time that constrained zero modes can give rise to dynamical symmetry breaking [9], and work continues on the role of zero modes and near zero modes in these theories [16].

To understand the properties of dimensionally reduced SYM-CS theory, we will need to review earlier results of similarly reduced SYM theory. This $\mathcal{N}=1$ SYM theory in $1+1$ dimensions is a stringy theory, in the sense that the low-mass states are dominated by Fock states with many constituents. As the size of the discrete superalgebra representation is increased, states with lower masses and more constituents appear [14,15]. In addition, this theory has a well-defined number of massless BPS states.

We begin by considering $\mathcal{N}=1$ supersymmetric CS theory in $2+1$ dimensions. The Lagrangian of this theory is

$$\mathcal{L} = \mathrm{Tr}\bigg(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}\gamma_\mu D^\mu \Psi + \kappa\bar{\Psi}\Psi + \frac{\kappa}{2}\epsilon^{\mu\nu\lambda}\bigg[A_\mu\partial_\nu A_\lambda + \frac{2i}{3}gA_\mu A_\nu A_\lambda\bigg]\bigg), \quad (1)$$

where κ is the CS coupling. The two components of the spinor $\Psi = 2^{-1/4}\binom{\psi}{\chi}$ are in the adjoint representation, and we will work in the large-N_c limit. The field strength and the covariant derivative are $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$ and $D_\mu = \partial_\mu + ig[A_\mu,]$, respectively. Supersymmetric variations of the fields lead to the supercharge components

$$Q^- = -i2^{3/4}\int d^2x\,\psi(\partial^+A^- - \partial^-A^+ + ig[A^+, A^-]),$$
$$Q^+ = -i2^{5/4}\int d^2x\,\psi(\partial^+A^\perp - \partial^\perp A^+ + ig[A^+, A^\perp]). \quad (2)$$

The supercharge fulfills the supersymmetry algebra,

$$\{Q^\pm, Q^\pm\} = 2\sqrt{2}P^\pm, \qquad \{Q^+, Q^-\} = -4P^\perp. \quad (3)$$

In order to express the supercharge in terms of the physical degrees of freedom, we use constraints which are obtained from the equations of motion to eliminate the nondynamical fields χ and A^-. In light-cone gauge, $A^+=0$, we reduce the theory dimensionally to two dimensions by setting $\phi = A_\perp$ and $\partial_\perp \to 0$ for all fields. This yields, from Eq. (2),

$$Q^- = 2^{3/4}g\int dx^-\bigg(i[\phi, \partial_-\phi] + 2\psi\psi - \frac{\kappa}{g}\partial_-\phi\bigg)\frac{1}{\partial_-}\psi. \quad (4)$$

To discretize the theory, we impose periodic boundary conditions on the boson and fermion fields alike and obtain expansions of the fields ϕ_{ij} and ψ_{ij} in terms of discrete momentum modes $A_{ij}(n)$ and $B_{ij}(n)$, respectively. The positive integers n correspond to discrete longitudinal momenta $k^+ = n\pi/L = nP^+/K$, where L is a longitudinal length scale, P^+ is the total momentum, and K is a positive integer that determines the resolution. (In DLCQ, K is known as the harmonic resolution [8].) The positivity of n guarantees that the number of partons in a Fock state is bounded by K. The discrete version of the CS part of the supercharge is

$$Q^-_{\mathrm{CS}} = \bigg(\frac{2^{-1/4}\sqrt{L}}{\sqrt{\pi}}\bigg)\kappa\sum_n \frac{1}{\sqrt{n}}[A^\dagger(n)B(n) + B^\dagger(n)A(n)]. \quad (5)$$

The continuum limit is the limit where $K\to\infty$.

Of the two contributions to the supercharge, Q^-_{SYM} and Q^-_{CS}, the former is imaginary and the latter real. Thus, the usual eigenvalue problem,

$$2P^+P^-|\varphi\rangle = \sqrt{2}P^+(gQ^-_{\mathrm{SYM}} + \kappa Q^-_{\mathrm{CS}})^2|\varphi\rangle = M_n^2|\varphi\rangle, \quad (6)$$

has to be solved by using fully complex methods to retrieve the mass eigenvalues M_n.

We retain the Z_2 symmetry associated with the orientation of the large-N_c string of partons in a state [17]. It gives a sign when the color indices are permuted

$$Z_2{:}A_{ij}(k) \to -A_{ji}(k), \qquad B_{ij}(k) \to -B_{ji}(k). \quad (7)$$

We reduce the numerical effort by using this symmetry to block diagonalize the Hamiltonian matrix. Eigenstates will be labeled by the Z_2 sector in which they appear.

It is interesting to note that the pure SYM supercharge is purely imaginary. Consequently, the lowest finite-dimensional representation of the superalgebra is four-dimensional, and there must be an exact fourfold mass degeneracy. On the other hand, the full SYM-CS supercharge is complex, and the lowest complex representation of the superalgebra is two-dimensional. Therefore the exact degeneracy only has to be twofold, which is what we find in our numerical results.

We have converted the mass eigenvalue problem, Eq. (6), to a matrix eigenvalue problem by introducing a discrete basis where the longitudinal momentum operator P^+ is diagonal. To obtain the spectrum of the SYM-CS theory, we diagonalize the Hamiltonian $P^- = (gQ^-_{\mathrm{SYM}} + \kappa Q^-_{\mathrm{CS}})^2/\sqrt{2}$.

The low-energy spectrum can be fit to $M^2 = M^2_\infty + b(1/K)$, where M^2_∞ and b are adjustable parameters. For a detailed discussion of these fits to the present theory, see Ref. [7]. The CS term in this theory effectively generates a mass proportional to the CS coupling. Therefore, we expect that the low-mass states will have only a few partons, reminiscent of QCD. This is interesting and important because it stands in stark contrast to $\mathcal{N} = 1$ SYM theory, which is very stringy and has a large number of low-mass states with a large number of partons.

From the structure of the Hamiltonian $P^- = (gQ^-_{\rm SYM} + \kappa Q^-_{\rm CS})^2/\sqrt{2}$, we expect that, as a function of g and κ, the spectrum of this theory will grow quadratically in both variables. At fixed g as a function of κ, the spectrum behaves exactly as expected [7]. In Fig. 1, we see that at fixed κ the masses of most of the states also grow quadratically in g. There are, however, a number of states which behave very differently, and it is on these states that we will focus our attention.

First, we need a detailed understanding of the spectrum at $g = 0$, because we will see that the theory has a new duality which relates the spectrum at $g = 0$ to that at $g = \infty$ at fixed κ. The Hamiltonian at $g = 0$ is the square of the CS supercharge, which is simply the Hamiltonian of free fermions and bosons with mass κ. In DLCQ, the free n-particle spectrum at resolution K is given by

$$M_n^2(K) = K\kappa^2\left(\sum_{i=1}^{n-1}\frac{1}{n_i} + \frac{1}{K - \sum_{i=1}^{n-1} n_i}\right), \tag{8}$$

where $n_i\pi/L$ is the longitudinal momentum of the ith particle. For example, at $K = 3$ the two-parton state has a mass squared of $M^2 = 4.5$ in units of κ^2, while at $K = 4$ there are two two-parton states with eigenvalues $M^2 = 4.0$ and $M^2 = 5.33$ in the same units. These states represent a discrete approximation of the two-particle continuum, which has its threshold at $M^2 = 4$ for even K and at $M^2 = 4/(1 - 1/K^2)$ for K odd. The threshold for the n-particle continuum is at $M^2 = n^2$. It is exactly reproduced at $K = nm$, with m a positive integer, and approaches this value in the continuum limit otherwise.

We will focus on the states whose energy remains almost constant with increasing $g\sqrt{N_c}$. These are the states that we classify as approximate BPS states. They cannot, of course, be true BPS states because the theory under consideration has an $\mathcal{N} = 1$ supersymmetry and can have only massless BPS states. Rather, these states are a reflection of the massless BPS states that we found in the pure SYM theory in two dimensions [14]. (They are also present in the $(2 + 1)$-dimensional SYM theory [12,13].)

In the region where $\kappa/g\sqrt{N_c}$ is small, we can understand the connection of these states to the massless BPS states of pure SYM by doing simple perturbation theory about the SYM theory. We rewrite the eigenvalue Eq. (6) by taking out a factor of g^2 to obtain

$$\left(\frac{1}{\sqrt{N_c}}Q^-_{\rm SYM} + \frac{\kappa}{g\sqrt{N_c}}Q^-_{\rm CS}\right)^2|\phi_n\rangle = \mathcal{E}_n|\phi_n\rangle, \tag{9}$$

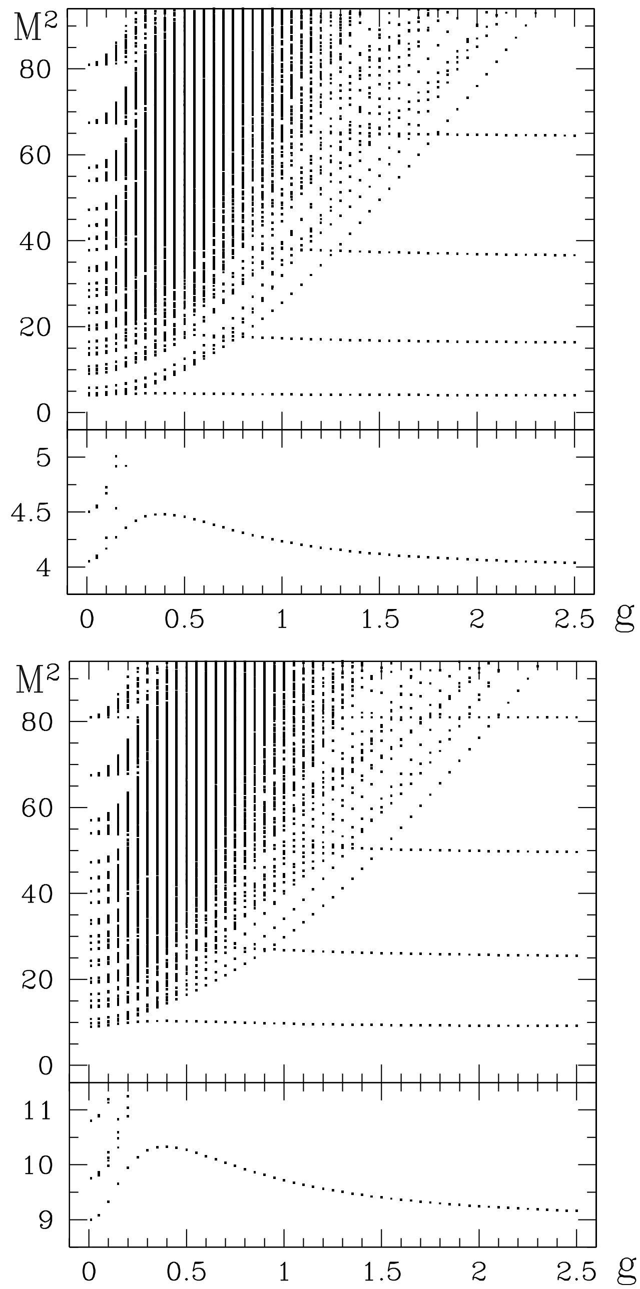

FIG. 1. Bosonic spectrum at $K = 9$ of the two-dimensional theory in units of κ^2 as a function of the gauge coupling $g\sqrt{N_c}$ at fixed Chern-Simons coupling κ for the (a) Z_2 even sector and (b) Z_2 odd sector. The enlarged plots are of the lowest states in the two sectors.

with $\mathcal{E}_n \equiv M_n^2/\sqrt{2}g^2N_cP^+$. We take $(Q^-_{\rm SYM})^2$ to be the unperturbed Hamiltonian and look for perturbations of the BPS states $|\psi_n\rangle$ of the pure SYM theory. Without loss of generality, we assume that they are bosons. (There is a set of massless BPS fermions as well.) The massless BPS states are approximate eigenstates of the number operator with "eigenvalues" $n = 2$ to K, and we use n to label the $(K - 1)$-fold degenerate bound states. (This is true to the numerical accuracy of this calculation, but we have reason to believe that there may be small additional corrections.) Of course, the zeroth order energy in the perturbation expansion vanishes. The important point is that the first-order corrections in the degenerate BPS subspace are determined by the matrix elements $\langle\psi_n|\{Q^-_{\rm SYM}, Q^-_{\rm CS}\}|\psi_m\rangle$, and they vanish as well. From this, we see that the leading order correction to the energy is of order $(\kappa/g\sqrt{N_c})^2$, and therefore the perturbative expansion for $\mathcal{E}_n$ can be written as

$$g^2N_c\mathcal{E}_n = \kappa^2\mathcal{E}_n^{(2)} + \mathcal{O}\left(\frac{\kappa^3}{g\sqrt{N_c}}\right). \tag{10}$$

Thus, the masses of these states are approximately independent of g for small $\kappa/g\sqrt{N_c}$. This is exactly what we see in Fig. 1. In particular, the detailed plots of the lowest states show a $1/g$ convergence of the mass towards the asymptotic value $\mathcal{E}_n^{(2)} = n^2$.

We return now to the new duality that these states exhibit. In the numerical calculations, we find that $\mathcal{E}_n^{(2)}$ is independent of the resolution K. Therefore, at large g^2N_c these states are exact threshold bound states, again independent of the resolution. The unperturbed massless BPS states $|\psi_n\rangle$ of the pure SYM theory are, to a good approximation, diagonal in particle number and have n partons. Therefore, it might not seem surprising that at $g^2N_c \to \infty$ the "BPS-like" states are threshold bound states. They are, however, threshold bound states with a special twist. Consider, for example, the simplest case with harmonic resolution $K = 3$. The discrete two-particle threshold is at $M^2 = 4.5$ in units of κ^2. However, at $g^2N_c \to \infty$ the "BPS-like" bound-state mass squared is $M^2 = 4.0$. This is the true threshold, and not the discrete threshold for resolution $K = 3$ where the calculation was performed. Therefore, at resolution $K = 3$ and $g^2N_c \to \infty$, this bound state is below the discrete threshold. From a detailed inspection of the mass matrix, we find that mixing between the dominant two-particle content and a very small three-particle content is essential for this to occur. The existence of this small three-particle content is a consequence of a theorem proven in Ref. [15], which states that a pure n-parton state cannot exist in two-dimensional supersymmetric field theories. For even values of the resolution, the discrete threshold is the correct threshold, and these "BPS-like" states are exact threshold bound states. For odd resolution, the discrete threshold approaches the correct threshold as we increase the resolution. The general statement of the new duality is that it relates the masses of these approximate BPS states at $g^2N_c = 0$ and at ∞ by

$$\lim_{K\to\infty}\mathcal{E}_n(g^2N_c = 0) = \lim_{g\sqrt{N_c}\to\infty}\mathcal{E}_n(g^2N_c). \tag{11}$$

In summary, we have found that, while $\mathcal{N} = 1$ supersymmetric theories cannot have local massive BPS states, they can have approximate BPS states whose masses are nearly independent of the Yang-Mills couplings. The existence of these states holds out the prospect of allowing one to extrapolate part of the massive spectrum of an $\mathcal{N} = 1$ supersymmetric theory into the strong-coupling regime in phenomenologically interesting theories.

These approximate BPS states occur because $\mathcal{N} = 1$ SYM theory has a set of massless BPS states. Using ordinary perturbation theory, we showed that at small $\kappa/g\sqrt{N_c}$ these approximate BPS bound states are a reflection of the massless BPS bound states of the SYM theory. Finally, we found that in the limit of infinite coupling g these states are threshold bound states which in turn leads to a duality relation of the spectrum at $g^2N_c = 0$ and ∞.

Since these BPS states are also present in the (2 + 1)-dimensional SYM theory, we expect to see their reflection in the (2 + 1)-dimensional CS theory. A study of this theory is underway.

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