Steep Sharp-Crested Gravity Waves on Deep Water

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A new type of steep two-dimensional irrotational symmetric periodic gravity wave with local singular point inside the flow domain is revealed on inviscid incompressible fluid of infinite depth. The speed of fluid particles in the vicinity of the crest of these waves is greater than their phase speed. Corresponding particle trajectories provide insight into how gravity waves overturn and break.

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A proper understanding of various wave phenomena on the ocean surface, such as the formation of breaking waves and whitecaps [1,2], solitary [3] and freak [4] waves, as well as modulation effects and instabilities of large amplitude wave trains [5,6] requires detailed knowledge of the form and dynamics of steep water waves. For the first time surface waves of finite amplitude were considered by Stokes [7]. Stokes conjectured that such waves must have a maximal amplitude (the limiting wave) and showed the flow in this wave to be singular at the crest forming a 120° corner (the Stokes corner flow). Much later, Grant [8] suggested that this singularity, for a wave that has not attained the limiting form, is located above the wave crest and forms a stagnation point with streamlines meeting at right angles. A strict mathematical proof of the existence of finite amplitude Stokes waves was given by Nekrasov [9]. Toland [10] proved that Nekrasov's equation has a limiting solution describing a progressive periodic wave train such that the flow speed at the crest equals the train phase speed, in the frame of reference, where fluid is motionless at infinite depth. Longuet-Higgins and Fox [11] constructed asymptotic expansions for waves close to the 120°-cusped wave (almost highest waves) and showed that the wave profile oscillates infinitely as the limiting wave is approached. Later, in [12], the crest of a steep irrotational gravity wave was theoretically shown to be unstable.

The purpose of the present work is to investigate the dynamics of gravity waves beyond the Stokes corner flow and to provide insight on the occurrence of deep water breaking. The traditional criterion for wave breaking is that horizontal water velocities in the crest must exceed the speed of the crest [1]. We give evidence for the existence of a family of two-dimensional irrotational symmetric periodic gravity waves that satisfy this criterion. A stagnation point in the flow field of these waves is inside the flow domain, in contrast to the Stokes waves of the same wavelength. This makes streamlines exhibit discontinuity with near-surface particles being jetted out from the flow. To all appearances, this effect is observed in the form of whitecapping of steep waves in the vicinity of their crests.

The original motivation was as follows: the Bernoulli equation is quadratic in velocity and admits two values of the particle speed at the crest. The first one corresponds to the Stokes branch of symmetric waves for which the particle speed at the crest is smaller than the wave phase speed. The opposite inequality takes place for the second branch, which might correspond to a new type of wave. In the second part of the Letter, we prove this numerically by using two different methods.

Consider a symmetric two-dimensional periodic train of waves propagating without changing its form from left to right along the x axis with constant speed c relative to the motionless fluid at infinite depth. The set of equations governing steady potential gravity waves on the surface of irrotational, inviscid, incompressible fluid is

$$\Phi_{\theta\theta} + \Phi_{yy} = 0, \qquad -\infty < y < \eta(\theta); \tag{1}$$

$$(c - \Phi_\theta)^2 + \Phi_y^2 + 2\eta = c^2, \qquad y = \eta(\theta); \tag{2}$$

$$(c - \Phi_\theta)\eta_\theta + \Phi_y = 0, \qquad y = \eta(\theta); \tag{3}$$

$$\Phi_\theta = 0, \qquad \Phi_y = 0, \qquad y = -\infty; \tag{4}$$

where $\theta = x - ct$ is the wave phase, $\Phi(\theta, y)$ is the velocity potential, $\eta(\theta)$ is the elevation of the free surface, and y is the upward vertical axis such that $y = 0$ is the still water level. We have chosen the units of time and length such that the acceleration due to gravity and wave number are equal to unity. Once the velocity potential is known, particle trajectories (streamlines) are found from the following differential equations:

$$\frac{d\theta}{dt} = \Phi_\theta(\theta, y) - c, \qquad \frac{dy}{dt} = \Phi_y(\theta, y); \tag{5}$$

in the frame of reference moving together with the wave.

As it follows from the Bernoulli equation (2), a solution may be not single valued in the vicinity of the limiting point. Indeed, the particle speed at the crest $q(0)$ is horizontal and is defined as follows:

$$\Phi_\theta[0, \eta(0)] = q(0) = c \pm \sqrt{c^2 - 2\eta(0)}, \tag{6}$$

$\eta(0)$ being the height of the crest above the still water level. Note that when using the term "crest" we mean the highest point of the wave surface that is located on the axis of symmetry of a wave. The "$-$" sign in (6) corresponds to the classical Stokes branch. The value $\eta_{\max}(0) = c^2/2$ corresponds to the Stokes wave of limiting amplitude. In this case, the particle speed at the crest is exactly equal to the wave phase speed: $q_{\max}(0) = c$. Taking into account both signs in expression (6), we assume that a second branch of solutions should exist apart from Stokes waves, at $\eta(0) < \eta_{\max}(0)$. The particle speed at the crest of a new gravity wave must be greater than c and has to increase from c to $2c$, while the wave height decreases from $\eta_{\max}(0)$ to 0.

Moreover, the mean levels of these two flows relative to the level $y = 0$ of still water may also be different:

$$\frac{1}{2\pi}\int_0^{2\pi} \eta^{(i)}(x)dx = \eta_0^{(i)}, \qquad i = 1, 2. \tag{7}$$

Thus, the existence of a second branch of solutions to the set of equations (1)–(4) does not contradict Garabedian's theorem [13] that gravity waves are unique if all crests and all troughs are of the same height because the latter was proved for flows with the same mean level. Furthermore, streamlines in a flow for which $q(0) > c$ seem to be discontinuous in the vicinity of the wave crest, whereas Garabedian's theorem deals with regular flows only.

To obtain numerical solutions we apply the method of truncated Fourier series and the collocation method using the spatial coordinates as independent variables. Herein we assume that it is possible to approximate discontinuous flows by continuous expansions [14].

The method of the Fourier approximations.—Let us introduce the complex function $R(\theta, y)$ such that

$$\Phi = -ic(R - R^*), \qquad \Psi = c(R + R^*), \tag{8}$$

where Ψ is the stream function and $*$ is the complex conjugate. Using the relations $\Phi_x = \Psi_y$, $\Phi_y = -\Psi_x$, the kinematic boundary condition (3) can be presented as follows:

$$\frac{d}{dx}[R(\theta, \eta) + R^*(\theta, \eta) - \eta(\theta)] = 0. \tag{9}$$

Approximate symmetric stationary solutions of Eqs. (1), (2), (9), and (4) are looked for in the form of the truncated Fourier series with real coefficients

$$R(\theta, y) = \sum_{n=1}^{N} \xi_n \exp[n(y + i\theta)]; \tag{10}$$

$$\eta(\theta) = \sum_{n=-M}^{M} \eta_n \exp(in\theta), \qquad \eta_{-n} = \eta_n; \tag{11}$$

where the Fourier harmonics ξ_n, η_n and the wave speed c are functions of the wave steepness A determined by the peak-to-trough height:

$$A = \frac{\eta(0) - \eta(\pi)}{2\pi} = \frac{2}{\pi}\sum_{n=0}^{[M/2]} \eta_{2n+1}, \tag{12}$$

square brackets designating the integer part. Substitution of expansions (10) and (11) into the dynamical and kinematic boundary conditions (2) and (9) [the Laplace equation (1) and boundary condition (4) are satisfied exactly] yields the following set of $N + M + 1$ nonlinear algebraic equations for the harmonics ξ_n, η_n, and the wave speed c:

$$\sum_{n_1=1}^{N} \xi_{n_1}(f_{n-n_1}^{n_1} + f_{n+n_1}^{n_1}) = \eta_n, \qquad n = \overline{1, N}; \tag{13}$$

$$c^2 \sum_{n_1=1}^{N} n_1 \xi_{n_1}\left(f_{n-n_1}^{n_1} + f_{n+n_1}^{n_1} - 2\sum_{n_2=1}^{N} n_2 \xi_{n_2} f_{n+n_1-n_2}^{n_1+n_2}\right) = \eta_n, \qquad n = \overline{0, M}; \tag{14}$$

where $f_n^{n_1}$ are the Fourier harmonics of the exponential functions $\exp[n_1\eta(\theta)]$:

$$f_n^{n_1} = \frac{1}{2\pi}\int_0^{2\pi} \exp[n_1\eta(\theta) - in\theta]d\theta, \qquad f_{-n}^{n_1} = f_n^{n_1}. \tag{15}$$

They were calculated using the fast Fourier transform (FFT). In addition to these equations, the connection (12) between the harmonics η_n and the wave steepness A should be taken into account.

The set of equations (13) and (14) was solved by Newton's iterations in arbitrary precision computer arithmetic. Since the nonlinearity over ξ_n and η_n is of different character (polynomial and exponential), M should be chosen greater than N to achieve good convergence. A different number of modes for the truncation of the Fourier series (10) and (11) was also used by Zufiria [15] in the framework of Hamiltonian formalism.

The method of collocations.—The harmonics ξ_n of expansion (10) can also be found in another way without expanding elevation into the Fourier series. In this approach, Eq. (2) and explicitly integrated Eq. (9) are to be satisfied in a number of the collocation points $\theta_j = j\pi/N$, $j = \overline{0, N}$, equally spaced over the half of one wavelength from the wave crest to the trough, similar to Rienecker and Fenton [16]. This leads to $2N + 2$ algebraic equations for the harmonics ξ_n, the values of the elevation η at the collocation points, and the wave speed c. To make the numerical scheme converge better a greater number of collocation points may be used in the dynamical boundary condition (2): $M = PN$, P being an integer.

The results of calculations and discussion.—The dependence $c(A)$ of the speed of steep gravity waves on their steepness is shown in Fig. 1. Along with the curves

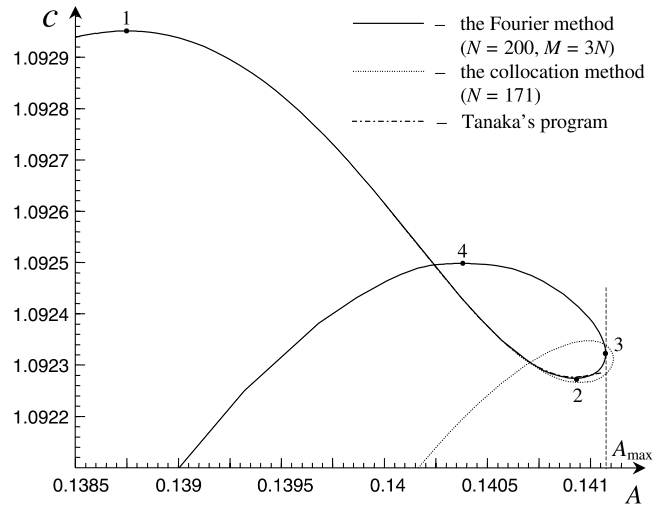

FIG. 1. The phase speed c of surface gravity waves versus their steepness A.

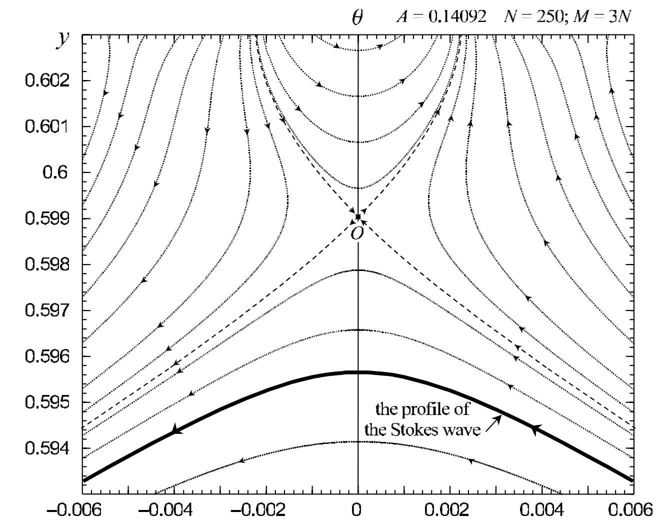

FIG. 2. The streamlines of the almost highest Stokes flow near the wave crest, extended outside the domain filled by fluid, in the frame of reference moving together with the wave.

obtained by the Fourier and collocation methods, we included high accuracy calculations of the Stokes branch obtained by Tanaka's program, where the method of inverse plane is used according to his paper [17]. In the plot, point 1 ($A = 0.138\,75$) is the maximum of wave speed, point 2 ($A = 0.140\,92$) is the relative minimum, and point 3 ($A = A_{\max} = 0.141\,074$) corresponds to the limiting steepness at N and M given. For greater N and M, $A_{\max} \gtrsim 0.141\,06$ is obtained. Note that less accurate calculations by the collocation method give a greater value of the limiting steepness, which is close to that reported by Schwartz [18].

The streamlines of the Stokes flow that has not attained its limiting form (almost highest flow) are extended outside the domain filled by fluid and are shown in Fig. 2 near the wave crest, in the frame of reference moving together with the wave. The profile $\eta(\theta)$ of a regular Stokes wave coincides all over the period with one of the streamlines calculated from (5). One can see from Fig. 2 that there is the stagnation point O above the wave crest with streamlines meeting at right angles in accordance with the results of Grant [8] and Longuet-Higgins and Fox [11]. There, such points are called singular of order one-half.

The key result of our numerical investigation is that besides the regular Stokes flow we revealed a singular potential flow with stagnation point O_1 of order one-half located inside the flow domain. As is seen from Fig. 3, the streamline coinciding with the wave profile $\eta(\theta)$ at some distance from the crest is discontinuous near the crest. Because of this we call such a wave and flow "irregular." At the same time, the function $\eta(\theta)$ defined by expansion (11) is continuous everywhere. Actually, such a continuous form of the profile for the irregular wave is the consequence of convergence in the mean of the Fourier series for a discontinuous function [14].

As wave steepness drops relative to the limiting value, the difference between flows in the crests of regular Stokes and irregular waves becomes stronger. On the contrary, as wave steepness tends to the limiting value, the singular points of both flows O and O_1 move towards the wave crest from above and below, respectively. Thus, we can assume that the 120° corner at the crest of the limiting Stokes wave is formed by merging these two singular stagnation points of order one-half. Therefore, the Stokes corner flow seems to be the superposition of a regular Stokes flow and a singular irregular flow.

Dependence of the wave speed of irregular waves on their steepness is represented in Fig. 1 by the branch 3-4. The speed of particles at the crests of the waves from this

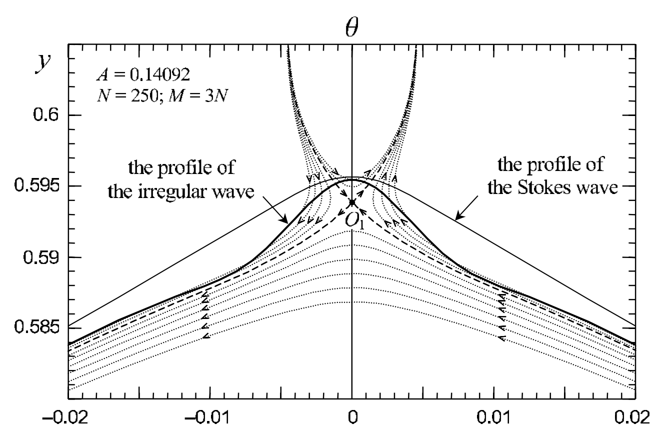

FIG. 3. The streamlines of the almost highest irregular flow near the wave crest, in the frame of reference moving together with the wave. The profile of the Stokes wave of the same steepness is also included for comparison.

TABLE I. The wave speed c for steep regular Stokes and irregular waves with wave steepness A calculated by the Fourier method at $N = 200, M = 3N$. The values for Stokes waves obtained by Tanaka's program are presented to estimate the accuracy of our calculations.

	Regular Stokes waves		Irregular waves
A	c	c_{Tanaka}	c
0.14	1.092 614 903 4	1.092 614 903 4	1.092 46
0.1406	1.092 337 63	1.092 337 749 9	1.092 49
0.140 92	1.092 274 2	1.092 276 839 2	1.092 42
	1.092 276 1[a]		1.092 43[a]
0.141	1.092 279 6	1.092 280 859 6	1.092 39
0.141 06	1.092 295[b]	1.092 285 104 7	1.092 355[b]

[a] $N = 250, M = 3N$.
[b] $N = 200, M = 4N$.

branch is greater than their phase speed. The values of the wave speed c for Stokes and irregular waves calculated by the Fourier method at different values of wave steepness are presented in Table I. High accuracy values for Stokes waves obtained by using Tanaka's program are also included for comparison. One can see that accuracy of the Fourier method for the Stokes branch gradually decreases as wave steepness increases up to the almost highest steepness $A = 0.141\,06$. The corresponding value of the wave speed has only five digits stabilized. While moving along the new branch accuracy becomes still less, and much greater N are needed to stabilize a greater number of digits. As a result, the loop in Fig. 1 has not yet stabilized at $N = 200$ and will enlarge with increasing N, the cross-section point with the Stokes branch moving to the left.

Irregular waves, which we found numerically using two independent methods, present a new type of singular gravity wave we looked for. In the present work, we are interested only in the existence of new stationary solutions and did not investigate their stability. It is known [17] that the total energy of Stokes waves attains the absolute maximum at the steepness $A = 0.1366$, which is much smaller than the limiting one. As wave steepness is increased further, regular Stokes waves become unstable due to superharmonic [17] or crest [12] instabilities. To all appearances, the obtained irregular flow with discontinuous streamlines is the result of developing these instabilities. This flow is characterized by the presence of the near-surface layer of fluid particles that are accelerated to velocities greater than the wave phase speed when approaching the crest. As a result, they form symmetric jets, which are apparently observed in the form of whitecaps near the crests of steep waves and are responsible for wave breaking. Verification of this assumption demands both taking into account surface tension and developing a more powerful numerical algorithm since the ones presented above become ineffective. We suppose that further investigation of obtained discontinuous irregular flows in more complicated systems would be useful for explaining appearance and evolution of many nonstationary processes such as wave overturning, formation of spilling and plunging breakers, bubble clouds, jets, etc. Finally, the fact of the existence of irregular flows follows from the Bernoulli equation (which represents the energy conservation law) and does not depend on depth, as follows from Eq. (6). We have recently proved this for a layer of finite depth [19].

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