Evaporation of a Packet of Quantized Vorticity

Carlo F. Barenghi and David C. Samuels
Mathematics Department, University of Newcastle, Newcastle NE1 7RU, United Kingdom
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We study the diffusion of a packet of quantized vorticity initially confined inside a small region. We find that reconnections fragment the packet into a gas of small vortex loops which fly away. The time scale of the process is in order-of-magnitude agreement with recent experiments performed in ^{3}He-B.

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The decay of superfluid turbulence at very low temperatures raises the fundamental question of the existence near absolute zero of an energy cascade from large to small scales. In ordinary turbulence, big eddies break up into smaller eddies, until the wave number is large enough that kinetic energy is dissipated by viscous forces. In superfluid turbulence, recent work indicates that it is generation of sound plays which plays the role of "sink" of kinetic energy [1]. It also appears that the generation of small scales occurs via creation of helical waves [2] of higher and higher wave numbers on the quantized vortex filaments (Kelvin waves), and via creation of small vortex loops [3]. The nonlinear mechanism behind both processes is vortex reconnection, either indirectly (reconnections create cusps which relax into large amplitude Kelvin waves) or directly (reconnections create small vortex loops).

The aim of this Letter is to show that the formation of small vortex loops is particularly important if the turbulence is not homogeneous, a case which is less investigated than homogeneous turbulence but is relevant to experiments at the lowest temperatures. Many experiments [4] have been performed in ^{4}He above 1.3 K, but little is known about lower temperatures. In the most relevant study [5] turbulence was produced by oscillating a grid at T as low as 20 mK ($0.01T_c$ where T_c is the critical temperature). Although there is no direct evidence, it is reasonable to expect that the vortex tangle was localized in the region of the grid. The measurements indicated that turbulence decayed in time. Direct studies of vortices in ^{3}He-B are typically done in a rotating cryostat [6], and an indirect observation of turbulence [7] has been confirmed only recently [8]. In this experiment turbulence was created by vibrating a wire at temperatures around $0.11T_c$. Again, we expect that turbulence was localized in the region of the vibrating wire. Additional wires were used to detect the Andreev reflection of quasiparticles from the tangle. The experimental setup could not tell unambiguously whether the growth time of the screening reflects the intrinsic temporal decay of the vorticity or the spatial evolution of the tangle away from the region in which it was created. Nevertheless, the authors argued from the temperature dependence that it is more likely that a spatial evolution of the tangle took place; typically, the spatial and temporal scales were 1 mm and 1 sec, respectively.

In both experiments, the lack of flow visualization makes the interpretation of the data difficult, so numerical simulations can give insight into the problem. Thus motivated, the simple questions which we address are the following: What is the fate of a tangle of quantized vortex loops initially confined in a small region? Can any physical processes cause the packet of quantized vorticity to diffuse out and spread in space?

In a classical Navier-Stokes fluid, diffusion of vorticity is caused by viscous forces. In the absence of viscosity (a perfect Euler fluid) vortex reconnections are not possible and the initial topology is frozen into the fluid; vortex loops distort but remain linked to each other, and conservation of helicity prevents the packet from spreading [9]. In a superfluid, reconnections are possible [10], so the question is whether the packet of quantized vorticity remains localized in the initial region or not, and if not, how fast it diffuses away. Additional motivation arises from the recent result [11] that the amount of twists and links of the turbulent tangle seems to be related to its ability to diffuse in the time scale under consideration.

We represent a vortex line as a space curve which, in the absence of friction, moves according to the classical Biot-Savart law [12]. By changing the discretization along the lines, we tested that the results do not depend on the reconnection procedure (details of the algorithm are in Ref. [13]). Since our model is incompressible, we have no transformation of kinetic energy (length of vortex line) into sound, an effect which can be studied using the Gross-Pitaevski model [1]. A small loss of length occurs due to the numerical reconnection procedure [3] but does not affect our results.

Using the above model, we have performed numerical experiments to determine the evolution of localized packets of vorticity. Although we used ^{4}He's parameter values, the results can be reinterpreted for ^{3}He by rescaling the units of time and length according to the different values of the quantum of circulation Γ. The vortex filament method, originally developed for ^{4}He, is equally valid for ^{3}He-B despite the much larger vortex core size than ^{4}He, because the length scales of the calculation (e.g., distance between discretization points on the same curve and distance between curves) are still much larger than the vortex core. Typically our initial condition consists of a given number N_0 of circular vortex rings, whose centers and orientations are randomly generated, initially confined in a sphere of radius S_0. Other initial conditions have been discussed in the related literature [14], notably that of random vortex network, but there is no reason to believe that they apply to our case, and we know too little of how a vibrating wire or grid generates quantized vorticity to be more realistic. Fortunately, it is known [12] that any simple configuration which is almost isotropic quickly evolves in a turbulent tangle independent of the initial state, and by numerically experimenting we found that our results do not depend on how we start the calculation. For example, we tried replacing many small circular rings with few longer Fourier knots (trefoil-like curves which wrap around themselves a few times before closing [15]). This changed drastically the initial topology, but the same results were found as for rings. Unlike previous numerical simulations of superfluid turbulence (performed with periodic boundary conditions or in channels with rigid walls), our calculations are carried out in an infinite volume. At each time, quantities which are useful to describe the vortex packet are the total length Λ, the number of loops N, the radius of the confining sphere S, the vortex line density $L = 3\Lambda/4\pi S^3$, the average intervortex distance $\delta = L^{-1/2}$, and the average vortex loop length in the packet D. These quantities depend on t, and we use the subscript zero to denote initial values.

The time evolution of the small vortex packet shown in Fig. 1 is typical. The initial vortex rings (here $N_0 = 20$) interact, become distorted, and reconnect. The evolution of the packet is determined by the balance between self-reconnections and reconnections between different loops [3]. During the initial coalescence phase the reconnections between different loops dominate and the number of separate loops decreases [$N(0.34) = 12$ in Fig. 1]. Following this comes an evaporation phase, when self-reconnections dominate. In this phase occasionally a self-reconnection generates a loop smaller than the average separation between the vortices. Because of its small size, the loop moves relatively fast, and, if created near the surface of the packet and oriented in the correct

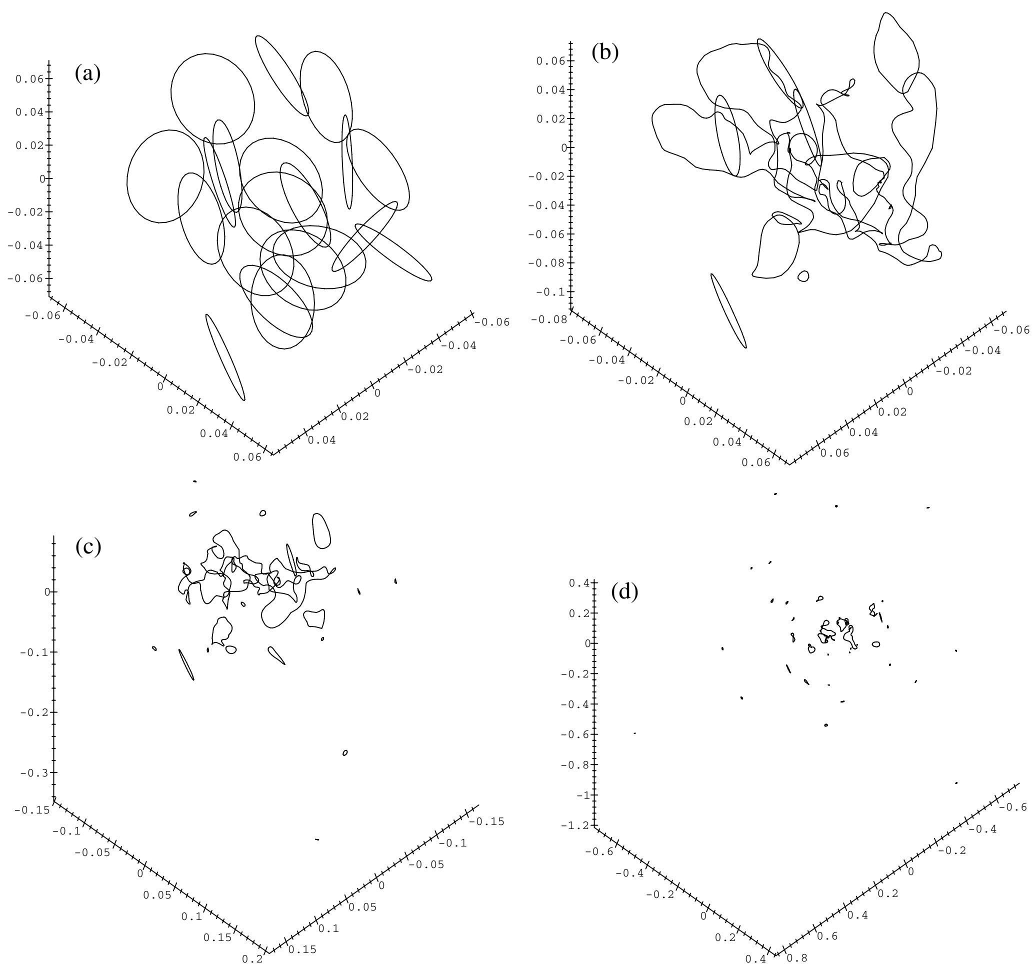

FIG. 1. Evolution [(a)–(d)] of a small vortex packet of quantized vorticity (data corresponding to the crosses in Fig. 3).

direction, it escapes from the packet, flying to infinity without further reconnections. Since each loop which escapes decreases the total vortex length in the packet, the average distance between vortices increases, thus increasing the probability that another loop escapes. The evaporation and escape phases distinguish homogeneous from inhomogeneous turbulence. In the former self-reconnections and reconnections between different loops come to a balance, and a steady state tangle forms [12]. In the latter, loops can quickly move out of the packet and then never reconnect again. A key ingredient of the effect is therefore the counterintuitive dispersion relation of vortex loops: unlike particles, the less energetic (smaller) they are, the faster they move. At this point N become constant [for example $N(2.2) = N(3.48) = 37$ in Fig. 1]. Self-reconnections dominate and the evaporation phase persists until the packet has expanded away. The speed v_R of an escaping loop of radius R can be estimated from the classical formula $v_R = (\Gamma/4\pi R)[\ln(8R/a) - 1/2]$, where a is the vortex core radius. In the actual experiments, of course, the volume is not infinite and the motion of isolated loops which escape terminates at the walls.

The interesting question is what determines the characteristic time scale for the vortex packet to evaporate. Since one important parameter of the problem is certainly the quantum of circulation, Γ, to obtain a time scale ℓ^2/Γ we must identify the relevant length scale ℓ. There are only two length scales in the problem: the size of the packet, S, and the average distance between the vortices, δ, which both change with time. For the sake of simplicity, hereafter we refer to the initial values S_0 and δ_0. The reason is that the definitions of S and δ at later times are somewhat arbitrary, as they are sensitive to the presence of small (fast) loops in a particular numerical calculation. The use of the initial values simplifies the analysis and lets us concentrate on the simple issue of whether we can predict the evolution of the packet given initial length Λ_0 and size S_0.

Figure 2 shows how a typical distribution of loop lengths changes with time. At $t = 0$ the distribution is on the 11th bin (all $N = 30$ loops have the same length $d = 0.067$ cm by construction). As time proceeds, the distribution moves to the 3rd bin (centered at $d \approx 0.013$ cm). Note the direct cascade from the initial peak at the right to the final peak at the left without creation of intermediate length scales. The position and height of the initial peak depends on the initial configuration (if we have few longer Fourier knots at the place of many small circular loops, the initial peak is smaller and more to the right). What is universal is the creation of the final left peak, which happens in all our simulations.

Now we analyze how the average loop length, $D = \langle d \rangle$, depends on t. For the sake of clarity, we normalize D using the maximum value $D_{\max}$ achieved in each particular run. If we plot $D/D_{\max}$ versus t we note an initial increase (coalescence) followed by a decrease (evaporation) which eventually remains constant as separate loops fly away in all directions (escape). The time scale for the packet to evaporate ranges in the interval 0.05 sec $< t <$ 2 sec, depending on the particular run. In Fig. 3 we plot $D/D_{\max}$ versus the scaled time t/τ where $\tau = \delta_0^2/\Gamma$. It is apparent that curves corresponding to the evolution of different packets now overlap, and evaporation takes place within the shorter interval $1.5 < t/\tau < 2$. If we plotted the same graph by scaling t with S_0 rather than δ_0 the curves would be very separate. Figure 3 therefore suggests that the characteristic time scale of evaporation is of the order of δ_0^2/Γ. Physically, τ represents the time scale of reconnections. In fact, from the quantization of vorticity $\oint \mathbf{v}_s \cdot d\mathbf{l} = \Gamma$, we estimate that the typical speed inside the packet is of the order of $v_s \approx \Gamma/\delta$; hence the typical reconnection time is of the order of $\delta/v_s = \delta^2/\Gamma = \tau$. Note that, since the distribution of values of δ is large, some filaments reconnect earlier, which is evident in Fig. 1 at the beginning of the run. The inset of Fig. 3 shows the evolution in space and time of different packets. Because of the above-mentioned difficulty with the definition of S, we use the more robust quantity S', defined as the radius of the sphere which contains *half* the total length. After the evaporation, the packet becomes a gas of loops which fly to infinity, so we expect $S' \approx vt$ where v is the speed of the typical loop. Using the formula for v_R we have, for $t > \tau$,

$$\frac{S'}{\delta_0} \approx \left(\frac{\mathcal{L}}{4\pi}\right)\left(\frac{t}{\tau}\right), \qquad (1)$$

where $\mathcal{L}$ is a term with a weak logarithmic dependence on δ_0. The inset of Fig. 3 confirms that S'/δ_0 and t/τ are proportional. The evolution of all packets are similar and collapse onto the same curve, as shown by the solid line which represents Eq. (1).

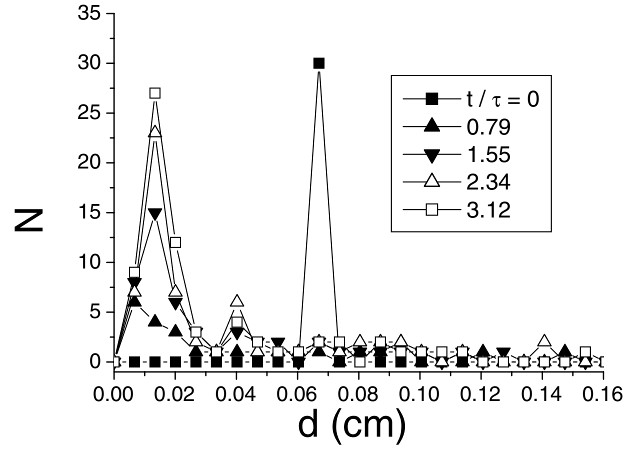

FIG. 2. Number N of loops of given length at different times corresponding to the upward triangles in Fig. 3. Note the direct cascade from the right peak to the left one (the small peak in the middle is due to some larger loops left at the center of the evaporated packet).

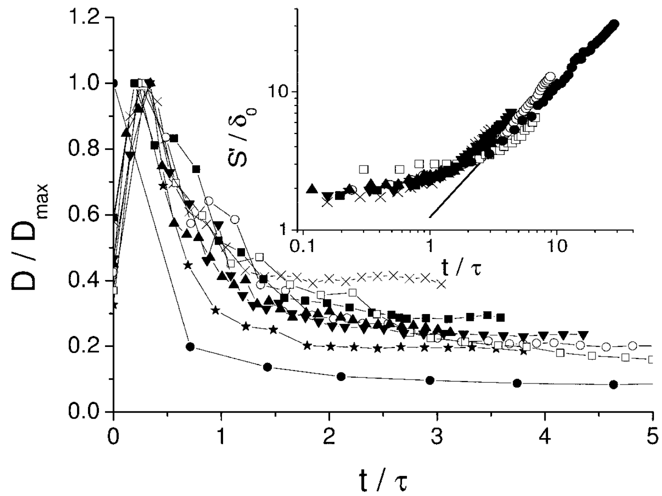

FIG. 3. Normalized average loop length D/D_{max} vs t/τ. The values of N_0, S_0 (cm), L_0 (cm^{-2}), and D_{max} (cm) for each run are the following: stars: 30, 0.090, 1252, and 0.38; crosses: 20, 0.090, 835, and 0.22; open circles: 30, 0.018, 31 296, and 0.059; upward triangles: 30, 0.045, 5007, and 0.139; solid squares: 25, 0.018, 26 080, and 0.042; downward triangles: 25, 0.027, 11 590, and 0.081; open squares: 60, 0.090, 2504, and 0.339. solid circles: 4, 0.018, 31 460, and 0.194 (in this case the initial condition consists of few long Fourier knots, so, unlike the other runs, reconnections immediately increase the number of separate loops and D is maximum at $t = 0$). The inset (S'/δ_0 vs t/τ) shows how the evolution of different packets scale together. The solid line shows Eq. (1).

We can now interpret the ^{3}He-B turbulence experiments [8]. In ^{3}He we have $\Gamma = h/2m_3 = 6.6 \times 10^{-4}$ cm^2/sec and $a \approx 10^{-6}$ cm. The tangle is created by a semicircular vibrating wire of diameter 0.3 cm, so we assume that the initial vortex packet has $S_0 \approx 0.1$ cm. We note that the time scale $S_0^2/\Gamma \approx 15$ sec is far too large to have relevance to what is observed. The number of vortices required to produce the observed barrier to the quasiparticles is estimated by the authors to correspond to a flow of order $v_s \geq 0.1$ cm/sec; hence, from $v_s = \Gamma/2\pi\delta_0$, we estimate $\delta_0 \geq 10^{-3}$ cm/sec, and we conclude that the vortex line density must be of the order of $L_0 \leq 10^6$ cm^{-2}. The characteristic time scale for the packet to evaporate into a gas of small rings is therefore of the order of $\tau = \delta_0^2/\Gamma \approx 1.5 \times 10^{-3}$ sec. The small loops fly away with speed $v_R \leq 0.4$ cm/sec estimated from the known formula for v_R since we know that $R \geq 10^{-3}$ cm. This result, that the velocity of expansion of the quantized vorticity is of the order of 1 mm/sec, is consistent with the observation that the vortex tangle spreads over the distance of 1 mm in the time of approximately 1 sec.

In conclusion, we have shown that a packet of quantized vorticity, initially localized in a small region, evaporates [16] and diffuses away as a gas of small vortex loops on the time scale of order $\tau = 1/L\Gamma$. Application of this scenario to the recent turbulent ^{3}He-B experiment yields order of magnitude agreement with the observed evolution. This cascade to small loops is similar to an idea originally proposed by Feynman [17]. To pursue this study in the context of ^{4}He it would be interesting to use the Gross-Pitaevskii model to determine whether small vortex loops radiate phonons.

Finally, our results should be of interest in fluid dynamics. First, we have found a peculiar form of diffusion in what is actually an inviscid fluid. Second, we have found a mechanism to transfer energy to small scales. Third, we have shown that, as far as helicity is concerned, the superfluid represents a different, third benchmark to study, besides the traditional Navier-Stokes and Euler fluids.

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[1] C. Nore, M. Abid, and M. E. Brachet, Phys. Rev. Lett. **78**, 3896 (1997); M. Leadbeater, T. Winiecki, D. C. Samuels, C. F. Barenghi, and C. S. Adams, Phys. Rev. Lett. **86**, 1410 (2001).
[2] W. F. Vinen, Phys. Rev. B **61**, 1410 (2000); B. V. Svistunov, Phys. Rev. B **52**, 3647 (1995); D. Kivotides, J. C. Vassilicos, D. C. Samuels, and C. F. Barenghi, Phys. Rev. Lett. **86**, 3080 (2001).
[3] M. Tsubota, T. Araki, and S. K. Nemirowskii, Phys. Rev. B **62**, 11751 (2000).
[4] S. R. Stalp, L. Skrbek, and R. J. Donnelly, Phys. Rev. Lett. **82**, 4831 (1999); J. Maurer and P. Tabeling, Europhys. Lett. **43**, 29 (1998).
[5] S. I. Davis, P. C. Hendry, and P. V. E. McClintock, Physica (Amsterdam) **280B**, 43 (2000).
[6] V. M. H. Ruutu, U. Parts, J. H. Koivuniemi, N. B. Kopnin, and M. Krusius, J. Low Temp. Phys. **107**, 93 (1997).
[7] C. Bauerle, Yu. M. Bunkov, S. N. Fischer, H. Godfrin, and G. R. Pickett, Nature (London) **382**, 322 (1996); D. I. Bradley, Phys. Rev. Lett. **84**, 1252 (2000).
[8] S. N. Fisher, A. J. Hale, A. M. Guenault, and G. R. Pickett, Phys. Rev. Lett. **86**, 244 (2001).
[9] H. K. Moffatt, J. Fluid Mech. **35**, 117 (1969).
[10] J. Koplik and H. Levine, Phys. Rev. Lett. **71**, 1375 (1993).
[11] C. F. Barenghi, R. L. Ricca, and D. C. Samuels, Physica (Amsterdam) **157D**, 197 (2001).
[12] K. W. Schwarz, Phys. Rev. B **31**, 5782 (1985).
[13] D. C. Samuels, in *Quantized Vorticity and Superfluid Turbulence*, Springer Lecture Notes in Physics, edited by C. F. Barenghi, R. J. Donnelly, and W. F. Vinen (Springer, New York, 2001).
[14] See, for example, the review by V. B. Eltsov, M. Krusius, and G. E. Volovik, cond-mat/9809125v3 (2000).
[15] L. H. Kauffman, in *Ideal Knots*, edited by A. Stasiak, V. Katritch, and L. H. Kauffman (World Scientific, Singapore, 1998), p. 364.
[16] Recently we learned of work by M. Tsubota, T. Araki, and W. F. Vinen, e-print cond-mat/0206253. Their Fig. 1 shows loops escaping from the tangle and confirms our findings.
[17] R. P. Feynman, in *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland, Amsterdam, 1955).