Eigenstates Ignoring Regular and Chaotic Phase-Space Structures

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We report the failure of the semiclassical eigenfunction hypothesis if regular classical transport coexists with chaotic dynamics. All eigenstates, instead of being restricted to either a regular island or the chaotic sea, ignore these classical phase-space structures. We argue that this is true even in the semiclassical limit for extended systems with transporting regular islands such as the standard map with accelerator modes.

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For a complete description of a quantum system the knowledge of its spectrum and eigenstates is necessary. It is a long-standing goal to obtain as much information about these fundamental quantum mechanical objects as possible from the properties of the corresponding classical system [1–3]. The semiclassical eigenfunction hypothesis [4,5] states that in the semiclassical limit almost all eigenstates are either regular or chaotic, i.e., their phase-space representations are concentrated on regions with either regular or chaotic classical dynamics, as illustrated in Figs. 1(b) and 1(c). Exceptions are of measure zero in the semiclassical limit and occur, e.g., at avoided level crossings where regular and chaotic states can hybridize. The semiclassical eigenfunction hypothesis is supported by convincing numerical and experimental data and forms the basis of the present understanding of spectral and dynamical properties of *finite* systems with a mixed phase space [7–13].

We study *extended* systems with transporting regular islands, i.e., with a chain of islands which are traversed sequentially in time (arrows in Fig. 1(a)). Transporting islands are ubiquitous in physical systems. In particular, they occur in spatially periodic systems with time periodic driving, such as Hamiltonian ratchets [13], atom-optic experiments [14,15], or in the paradigmatic model of quantum chaos–the standard map [16], where they are called accelerator modes.

In view of the common confidence in the semiclassical eigenfunction hypothesis and its importance, our result comes as a considerable surprise: The very concept of regular and chaotic states does not apply in the presence of transporting regular islands. Instead, we find that all eigenstates spread over regular islands as well as the chaotic sea of classical phase space. An example of such an "amphibious" eigenstate can be seen in Fig. 1(a). We argue that even in the semiclassical limit *all* eigenstates will ignore classical phase-space structures as soon as transporting islands are present and we will discuss a number of questions raised by this failure of the semiclassical eigenfunction hypothesis for extended systems.

In order to allow for an efficient calculation of quantum eigenstates for systems with transporting islands we consider one-dimensional kicked Hamiltonians

$$H = T(p) + V(x) \sum_n \delta(t - n), \qquad (1)$$

with a classical time-evolution described by the map

$$x_{t+1} = x_t + T'(p_t), \qquad p_{t+1} = p_t - V'(x_{t+1}).$$

We are interested in spatially periodic dynamics of period 1 and consider the simple situation where also $T(p + 1) = T(p)$ with a cyclic momentum variable, $p \equiv p + 1$. This ensures that the phase-space averaged current is zero. We choose the functions $V'(x)$ and $T'(p)$ such that they give rise to a large regular island transporting in x

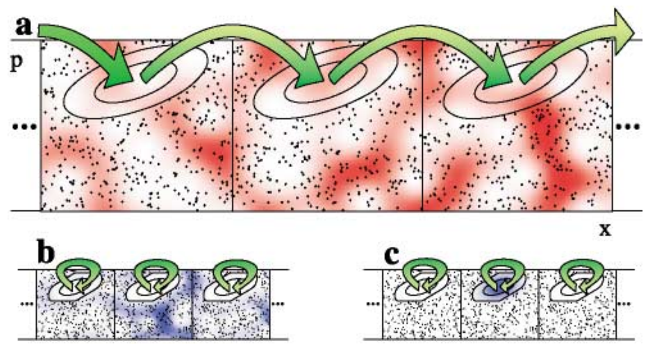

FIG. 1 (color). Spatially periodic phase-space portraits (black dots) of the system given by Eq. (1) [6] showing a large island of regular motion and a chaotic sea in each spatial unit cell with $p \in [-1/2, 1/2]$. Eigenstates of the corresponding quantum system are projected onto phase space (colored density). (a) The regular island is transporting as indicated by the arrows. All eigenstates, such as the one shown, spread over the regular islands as well as the chaotic sea. These amphibious eigenstates ignore the classical structures even though Planck's constant h is smaller than the size of the island. For comparison we show in (b) and (c) a system with regular islands of similar size that are not transporting [6]. Two different types of eigenstates—(b) chaotic and (c) regular—exist, in agreement with the semiclassical eigenfunction hypothesis.

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direction [6] as can be seen in Fig. 1(a). This makes the semiclassical regime, where the island area is bigger than Planck's constant h, numerically more easily accessible.

The quantum mechanical time-evolution operator for one period,

$$\hat{U}|\phi(t)\rangle = |\phi(t+1)\rangle, \tag{2}$$

is given by $\hat{U} = \exp[-2\pi i T(\hat{p})/h_{\text{eff}}] \exp[-2\pi i V(\hat{x})/h_{\text{eff}}]$, where h_{eff} is dimensionless and irrational, denoting the ratio of Planck's constant h to the phase-space area of a unit cell. The irrationality of h_{eff} makes this quantum system aperiodic and excludes the applicability of Bloch's theorem which otherwise would lead to spatially periodic eigenstates. We thereby model the experimentally relevant situation of spatially periodic systems with some disorder [17]. The eigenstates $|\psi\rangle$ of $\hat{U}$,

$$\hat{U}|\psi\rangle = e^{-i\epsilon}|\psi\rangle, \tag{3}$$

are the objects of our interest.

These eigenstates can be represented in phase space (Husimi representation [18]) and are compared with the classical phase-space portrait in Fig. 1. In contrast to the semiclassical eigenfunction hypothesis we find that all eigenstates are amphibious in all spatial unit cells, living on regular islands as well as in the chaotic sea. For comparison, we present in Figs. 1(b) and 1(c) a system which is equivalent in terms of phase-space portrait, island size, and effective Planck's constant h_{eff}, but with nontransporting islands. In this case the eigenstates can be categorized as chaotic (Fig. 1(b)) or regular (Fig. 1(c)) in agreement with the semiclassical eigenfunction hypothesis. This demonstrates that the mere presence of *transporting* regular islands leads to a failure of the semiclassical eigenfunction hypothesis.

While Fig. 1 shows the local structure of the eigenstates on the scale of a few unit cells, amphibious eigenstates are also distinguished from chaotic eigenfunctions from a global perspective. We find that they are localized with a much larger localization length than without transporting islands (Fig. 2), as anticipated from previous studies of wave packet dynamics [19–22]. This localization length of the eigenfunctions is determined by the tunneling out of the island: A wave packet initialized at the center of a regular island initially follows the classical dynamics (Fig. 3). Its weight inside the island, however, decays as $P(t) = \exp(-\gamma t)$, where γ is the tunneling rate between the regular island and the chaotic sea [19]. During the time scale $1/\gamma$ of this decay the wave packet is transported with the velocity $v = 1$ of the island. We thus estimate the localization length as $\lambda = v/\gamma$. We determined the tunneling rate $\gamma = 9.885\ 10^{-4}$ from the wave packet dynamics. The localization length of the amphibious eigenstates agrees well with the length $\lambda = 1011$ estimated from this tunneling rate, as can be seen in Fig. 2. The scaling of the tunneling rate $\gamma \sim$

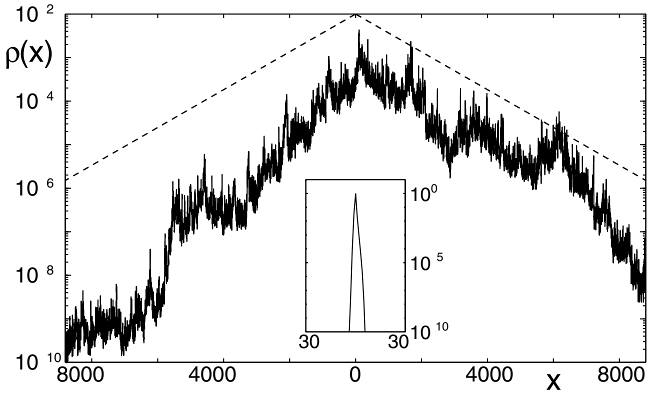

FIG. 2. The weight of the amphibious eigenstate from Fig. 1(a) in each unit cell, $\rho(x) = \int_{x-1/2}^{x+1/2} dx' \, |\psi(x')|^2$, shows an overall exponential decay with strong fluctuations. The localization length is several orders of magnitude bigger than in the case of the system of Fig. 1(b) with nontransporting islands (inset). It is quite well estimated by $\exp(-|x|/\lambda)$ (dashed line), with $\lambda = v/\gamma$ determined from the tunneling rate γ out of the regular island with velocity $v = 1$.

$\exp(-C/h_{\text{eff}})$ [19], with some constant C, yields a localization length $\lambda \sim \exp(C/h_{\text{eff}})$ that increases exponentially with decreasing h_{eff}. As a numerical computation of eigenstates requires a system size of several localization lengths, this scaling implies that approaching the semiclassical limit of small h_{eff} in the presence of transporting islands entails an enormous increase in numerical effort. Note that in the nontransporting case, $v = 0$, the above estimate for the localization length breaks down and the usual dynamical localization with $\lambda \sim h_{\text{eff}}^{-1}$ takes place.

An intuitive understanding of why transporting islands lead to amphibious eigenstates can be gained in the following way: The quantum mechanical continuity equation of a state φ is

$$\frac{\partial}{\partial t}|\varphi(x,t)|^2 + \frac{\partial}{\partial x}J_{\varphi}(x,t) = 0, \tag{4}$$

where J_{φ} is the probability current of φ [23]. Integration over one temporal period leads to

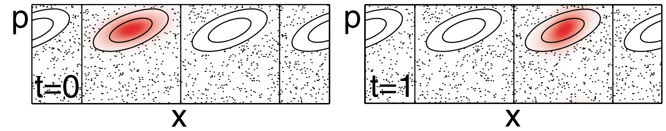

FIG. 3 (color online). Quantum evolution of a wave packet (gray shade) from $t = 0$ (left) to $t = 1$ (right) for the system of Fig. 1(a) where classically a regular island moves one unit cell to the right in one time period. Although the underlying amphibious eigenstates ignore the classical phase-space structure they conspire such that the initial dynamics of wave packets follows the semiclassical expectation.

$$|\varphi(x, t+1)|^2 - |\varphi(x, t)|^2 = -\frac{\partial}{\partial x}\int_t^{t+1} dt J_\varphi(x, t). \quad (5)$$

The left-hand side vanishes for eigenstates $|\psi\rangle$ of the time-evolution operator over one period, Eq. (3). Thus their temporally averaged current is spatially constant. Moreover, this constant is zero, because all eigenstates are localized and thus have a vanishing overall current. Therefore, the probability current

$$\int_t^{t+1} dt J_\psi(x, t) \equiv 0, \quad (6)$$

vanishes for all x. We can use this condition to infer the local structure of the eigenstates in the semiclassical regime, where the quantum probability current J_ψ is close to the classical current. In particular, we can immediately exclude regular eigenstates that are mainly concentrated on the transporting islands and thus would show a considerable current. Similarly, chaotic eigenstates can be excluded, as the chaotic sea also shows a nonzero current in the opposite direction [13]. The amphibious eigenstate presented in Fig. 1(a), however, is distributed uniformly over the phase-space regions with opposing classical currents. It fulfills the requirement of a vanishing current of Eq. (6) because the phase-space averaged classical current is zero. Hence we can conclude that even in the semiclassical limit *all* eigenstates spread simultaneously over regular and chaotic regions of the phase space in all spatial unit cells.

It is illuminating to point out where the arguments leading to the semiclassical eigenfunction hypothesis fail in the case of transporting islands. In the semiclassical limit the tunneling rate γ between the center of a regular island and the chaotic sea decreases to arbitrarily small values. Perturbation theory in γ can be used to calculate the eigenstates provided that the dimensionless coupling is small, $\gamma/\Delta \ll 1$, where Δ denotes the mean level spacing of those unperturbed states that will be coupled by tunneling. In extended systems it is crucial to understand the dependence of Δ on the system size L: In the absence of classical transport the unperturbed regular and chaotic states are localized, and the tunneling out of the island may couple only groups of states that are spatially close. In particular, the number of these states and their mean level spacing is fixed for given h_{eff} and does not depend on the system size, leading to $\Delta \sim h_{\text{eff}}$. Using $\gamma \sim \exp(-C/h_{\text{eff}})$ we see that perturbation theory is applicable ($\gamma/\Delta \ll 1$) for small h_{eff}. Hence the semiclassical eigenfunction hypothesis is valid. In the presence of transporting islands, however, the unperturbed regular states are extended and all of them will be coupled to all chaotic states. The mean level spacing of the unperturbed states is now inversely proportional to the system size, $\Delta \sim h_{\text{eff}}L^{-1}$, and perturbation theory breaks down ($\gamma/\Delta \approx 1$) as soon as $L \sim h_{\text{eff}}/\gamma$. In particular, we conclude that for an infinite system, transporting regular and chaotic regions are strongly coupled even in the semiclassical limit [24]. From the breakdown of perturbation theory alone, however, we cannot conclude the locally amphibious nature of eigenstates which we find numerically and deduce from the condition of vanishing current Eq. (6).

What do eigenstates look like if one has several transporting islands with different velocities? Let us illustrate this question with the important case of a symmetric phase space containing equivalent islands transporting in opposite directions (arrows in the lower inset of Fig. 4), as in the standard map with accelerator modes [16]. In this case regular eigenstates that are supported simultaneously by both islands would fulfill the condition of vanishing current, Eq. (6). However, due to the breakdown of the perturbation theory, we do not expect such states. A numerical investigation of the eigenstates reveals a structure which is consistent with all our arguments but surprisingly complicated: We find that all eigenstates are of a mixed type—in some unit cells they are concentrated on both regular islands, in some they spread over the chaotic sea, and in some unit cells they are amphibious (Fig. 4). This unexpected finding calls for further exploration.

Although the amphibious eigenstates disregard the classical phase-space structure, our results do not contradict quantum-classical correspondence. For example, the wave packet of Fig. 3 follows the classical expectation for arbitrarily long times $t \to \infty$, if the limit $h_{\text{eff}} \to 0$ is taken first. For eigenstates, however, the order of these limits is reversed since, according to the energy-time

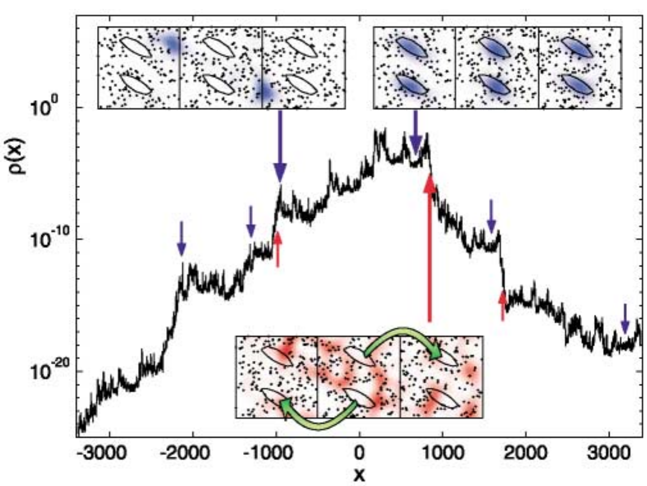

FIG. 4 (color). A typical eigenstate for a system with a symmetric phase space and islands transporting in opposite directions (arrows in lower inset) [25]. Several regions are projected onto phase space (insets) demonstrating the mixed type of this eigenstate, with some regions being mainly regular, some mainly chaotic, and some amphibious. Each type appears at many locations of the eigenstate (short arrows).

uncertainty, a precisely defined energy corresponds to infinite time for any nonzero $h_{\rm eff}$. As a consequence eigenstates may differ from the semiclassically expected states which are called quasimodes [26,27]. A famous example is the symmetric double-well potential, where eigenstates must be symmetric or antisymmetric and thus live in both wells, while classically the dynamics is restricted to one of the wells. This is a very special situation, however, which is destroyed already by a weak perturbation of the symmetry of the double-well potential. In contrast, amphibious eigenstates are a nontrivial example for the failure of classical intuition about eigenstates and for extended systems with transporting islands they are a generic phenomenon.

The failure of the semiclassical eigenfunction hypothesis and the appearance of amphibious eigenstates raise further questions: transporting island chains of arbitrary length exist even in systems with a finite phase space, such as in the neighborhood of the boundary circle enclosing a regular island. Will there be amphibious eigenstates? The statistical properties of chaotic eigenstates [3] can be seen in conductance measurements on quantum dots [28] and deviations from the quantum ergodic limit inside the chaotic component caused by, e.g., scars [29], superscars [30], or hierarchical states [11] are thus experimentally accessible. But what are the statistical properties of amphibious eigenstates? Are they similar to those of chaotic eigenstates or will there be specific correlations caused by the regular islands?

The recently developed techniques to observe atom dynamics in optical lattices [14,15] are perfectly suited for the experimental study of amphibious eigenstates. In such systems wave packets can be prepared on selected points in phase space, e.g., on the center of an island, and their long-time dynamics can be studied. A measurement of the phase-space distribution would reveal that the asymptotic wave packet is uniformly distributed over phase space, independent of the initial preparation. This is in sharp contrast to a system with nontransporting islands and would be a clear experimental signature of amphibious eigenstates.

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