Comment on "Indispensable Finite Time Corrections for Fokker-Planck Equations from Time Series Data"

Ragwitz and Kantz [1] propose a correction to a method for the reconstruction of Fokker-Planck equations from time series data. In [2–5] a method was presented which directly applied the mathematical definitions of the drift $D^{(1)}$ and diffusion $D^{(2)}$ terms [6] for an estimate from time series. Here different moments of conditional probability densities (pdf) for finite step sizes Δ in the limit $\Delta \to 0$ have to be estimated. Ragwitz and Kantz state that previous results have not been checked and that indispensable finite time step Δ corrections have to be employed for reliable estimates of $D^{(2)}$. We want to add the following comments.

Ragwitz and Kantz base their investigation on an estimate of the finite time conditional probability in terms of a Gaussian, Eq. (7) of their paper. There is, however, no reason that for finite Δ the conditional pdf is Gaussian. The exact expressions for the conditional moments up to the order Δ^2 can be unambiguously derived from the Fokker-Planck equation [7]:

$$\begin{aligned}\langle x - x_0 | x_0 \rangle &= \Delta D^{(1)} + \tfrac{1}{2}\Delta^2[D^{(1)}(D^{(1)})' + D^{(2)}(D^{(1)})''] \\ &\quad + O(\Delta^3) \\ \langle (x - x_0)^2 | x_0 \rangle &= 2\Delta D^{(2)} + \Delta^2[(D^{(1)})^2 + 2D^{(2)}(D^{(1)})' \\ &\quad + D^{(1)}(D^{(2)})' + D^{(2)}(D^{(2)})''] \\ &\quad + O(\Delta^3), \end{aligned} \tag{1}$$

For $\langle (x - x_0)^2 | x_0 \rangle$ the ansatz (7) of [1] neglects the last two terms which are important for processes involving multiplicative noise as it is the case for turbulence. This remark especially applies to the wind data presented in [1]. The validation of their method based on a Langevin [Eq. (9)] works only since purely additive noise is considered.

For turbulence Ragwitz and Kantz claim to obtain remarkable correction, as shown in their Fig. 6. In our approach [5] we estimate the diffusion term using the limit $\Delta \to 0$ yielding a dependency which can be approximated by a low order polynomial. In order to improve this estimate, the coefficients of this polynomial have been varied such that the solution of the corresponding Fokker-Plank equation yields an accurate representation of the measured one. In Fig. 1 we present a case where a large correction of the $\Delta \to 0$ estimation of $D^{(2)}$ had to be introduced (usually corrections are much smaller). For finite values of Δ the estimated values of $D^{(2)}$ clearly differ from the limiting case $\Delta \to 0$. Estimations with different "correction" terms for $D^{(2)}$ and for finite Δ values may fake large corrections values. Taking the limit $\Delta \to 0$, these deviations vanish within the error.

The range of Δ which can be taken for the estimate of $D^{(2)}$ must be chosen carefully in order to ensure that the Markovian property holds; see [5]. Since for each estimated value of $D^{(2)}$ a finite number of data points is used, a statistical error can be estimated for $D^{(2)}$ (cf. Fig. 1). These errors naturally increase considerably for large values of x (compare Fig. 6 [1] and Fig. 13 [5]).

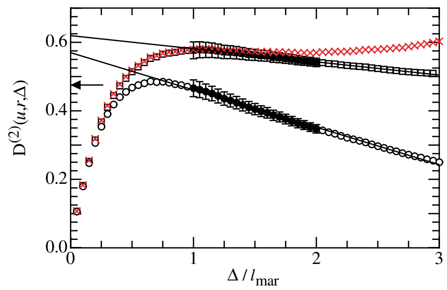

FIG. 1 (color online). Δ dependence of $D^{(2)}(u = \sigma_\infty, r = L/2, \Delta)/\sigma_\infty^2$ for different correction terms: squares without correction term, circles with $[(D^{(1)})^2]$, and crosses with the correction term of [1]. L denotes the integral length, and σ_∞ the rms of the velocity increments at large scales. Only for $\Delta/l_{mar} > 1$ do Markovian properties hold, and are estimations of $D^{(2)}$ senseful. The optimal value of $D^{(2)}$ based on verifications is indicated by an arrow. For further detail see [5].

To conclude, a deeper understanding of finite time correlations are of interest and may be used to improve the estimation of drift and diffusion terms. Up to now the best way for estimating diffusion coefficients is to combine a nonparametric estimate for $\Delta \to 0$ with a functional ansatz, i.e., a suitable polynomial ansatz. Refining the estimates of the coefficients by parametric methods leads to improved results by a comparison of measured and calculated conditional pdfs at finite Δ.

R. Friedrich,[1] Ch. Renner,[2] M. Siefert,[2] and J. Peinke[2]
[1]Universität Münster
Institut für Theoretische Physik
D-48149 Münster, Germany
[2]Carl-von-Ossietzky-Universität Oldenburg
FB-8 Physics, D-21111 Oldenburg, Germany

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[1] M. Ragwitz and H. Kantz, Phys. Rev. Lett. **87**, 254501 (2001).
[2] R. Friedrich and J. Peinke, Phys. Rev. Lett. **78**, 863 (1997).
[3] S. Siegert, R. Friedrich, and J. Peinke, Phys. Lett. A **243**, 275–280 (1998).
[4] R. Friedrich *et al.*, Phys. Lett. A **271**, 217 (2000).
[5] Ch. Renner, J. Peinke, and R. Friedrich, J. Fluid Mech. **433**, 383 (2001).
[6] A. N. Kolmogorov, Math. Ann. **140**, 415–458 (1931).
[7] The Taylor expansion of the conditional expectation of a function $f(x)$ is simply $\langle f(x(\Delta)) | x \rangle = \sum \frac{\Delta^n}{n!}(L^\dagger)^n f(x)$, where $L^\dagger$ is the adjoint Fokker-Planck operator.

