

Energy Spectrum of Superfluid Turbulence with No Normal-Fluid Component

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(Received 22 January 2002; revised manuscript received 24 June 2002; published 16 September 2002)

The energy of superfluid turbulence without the normal fluid is studied numerically under the vortex filament model. Time evolution of the Taylor-Green vortex is calculated under the full nonlocal Biot-Savart law. It is shown that for $k < 2\pi/l$ the energy spectrum is very similar to the Kolmogorov's $-5/3$ law which is the most important statistical property of the conventional turbulence, where k is the wave number of the Fourier component of the velocity field and l is the average intervortex spacing. The vortex length distribution converges to a scaling property reflecting the self-similarity of the tangle.

DOI: 10.1103/PhysRevLett.89.145301

PACS numbers: 67.40.Vs, 67.40.Bz

Particular attention has been focused recently on the similarity between superfluid turbulence and conventional turbulence [1–3]. Early work of the superfluid turbulence has been concerned with the counterflow where the normal fluid and superfluid flow oppositely [4], having no classical analog. However Stalp *et al.* studied recently the superfluid turbulence produced by the towed grid, thus finding the similarity between the superfluid turbulence and the conventional turbulence above 1.4 K [2]. They observed indirectly the Kolmogorov law which is one of the most important statistical properties of the conventional turbulence. This is understood by the idea that the superfluid and the normal fluid are likely to be coupled together by the mutual friction between them and to behave like a conventional fluid [1,5]. Since the normal fluid is negligible at mK temperatures, an important question arises: even free from the normal fluid, is the superfluid turbulence still similar to the conventional turbulence or not?

As the physical model to describe the vortex dynamics in superfluid He at very low temperatures, two types of models are well known: the Gross-Pitaevskii (GP) equation which describes the motion of a weakly interacting Bose condensate, and the vortex filament model governed by the incompressible Euler dynamics. The former reduces to the Euler vortex filament model when variations of the wave function over scales of the order of the superfluid healing length are neglected. The GP equation includes such complicated compressible effects as the radiation of sound from the vortex lines [6,7], the vortex-sound interactions, etc. In order to consider the intrinsic property of superfluid turbulence in a simpler situation, we study the energy spectrum of the 3D velocity field induced by the vortex tangle in the absence of the normal fluid under the vortex filament model.

The energy spectrum of the vortices in superfluid was numerically calculated by other authors. Nore *et al.* studied the energy spectrum of the decaying superfluid turbulence by using the GP equation and found the transient spectrum for small k has the Kolmogorov law [3].

However, at a late stage some complicated compressible effects become dominant. On the other hand, an advantage of the vortex filament model compared with the GP equation is the following: First, this model enables us to calculate the energy spectrum free from such complicated effects. Second, some physical quantities, e.g., a total line length of vortices and a vortex length distribution, can be calculated easily, so that the relation between the statistical property in the wave number space and the self-similarity of the vortex tangle in the real space can be discussed. Under the vortex filament model, the energy spectra were reported on the Kelvin wave cascade [8,9], which was limited to a few vortices, not a dense tangle. Thus the calculation of the energy spectrum of a dense tangle under the vortex filament model is expected.

For superfluid ^4He , the vortex filament model is very useful, because the vortex core radius $a_0 \sim 10^{-8}$ cm is microscopic and the circulation $\kappa = 9.97 \times 10^{-4}$ cm²/sec is fixed by quantum constraint. Since Helmholtz's theorem for a perfect fluid states that the vortex moves with the velocity produced by itself, the dynamics is governed by the Biot-Savart law [10]. The velocity field due to the Biot-Savart law is divided into two parts: one is the localized induction field determined by a local curvature of vortex line, and the other is the nonlocal field obtained by carrying out the integral of the Biot-Savart law along the rest of the filament. When the dynamics of a dense vortex tangle is calculated numerically, the nonlocal velocity field is usually neglected (the localized induction approximation) [10,11]. However, the dynamics in this work is calculated by the full nonlocal Biot-Savart law, because the long range effects can be important on this problem. A vortex filament is represented by a single string of points at a distance $\Delta\xi$ apart. When two vortices approach within $\Delta\xi$, it is assumed that they are reconnected. The computational sample is taken to be a cube of size $D = 1.0$ cm. This calculation assumes the walls of the cube to be smooth and takes account of image vortices so that the boundary condition may be satisfied. This calculation of the dynamics is made by the

resolution $\Delta\xi = 1.83 \times 10^{-2}$ cm and $\Delta t = 4.0 \times 10^{-3}$ sec.

The energy spectrum $E(k)$ is defined as $E = \int_0^\infty dk E(k)$, where E is the kinetic energy per unit mass and k is the wave number of the velocity field. In our previous papers we derived the energy spectrum under the vortex filament model [12]:

$$E(k) = \frac{\kappa^2}{2(2\pi)^3} \int \frac{d\Omega_k}{|\mathbf{k}|^2} \iint d\xi_1 d\xi_2 \times s'(\xi_1) \cdot s'(\xi_2) e^{-k \cdot [s(\xi_1) - s(\xi_2)]}, \quad (1)$$

where $d\Omega_k$ denotes the volume element $k^2 \sin\theta_k d\theta_k d\phi_k$ in spherical coordinates. Here a vortex filament is represented by the parametric form $s = s(\xi, t)$, where s refers to a point on the filament, the prime denotes differentiation with respect to the arc length ξ , and the integration is taken along the filament. The energy spectrum $E(k)$ is calculated for the vortex configuration $s(\xi)$ obtained by the simulation of the dynamics.

Figure 1 shows the decay of the vortex tangle without the mutual friction [13]. As the initial configuration of vortices, we use the Taylor-Green vortex [3] [Fig. 1(a)]. These initial vortices are highly polarized. However, through the chaotic dynamics which includes lots of reconnections, the vortices become a homogeneous and isotropic vortex tangle [Figs. 1(b)–1(d)].

First we discuss the transient behavior of the k dependence of the energy spectrum $E(k)$. The energy spectra calculated from each configuration in Fig. 1 are shown in Fig. 2. It is shown that the slope is changed about at $k = 2\pi/l$, where l is the average intervortex spacing. The energy spectrum for $k > 2\pi/l$ has k^{-1} behavior which

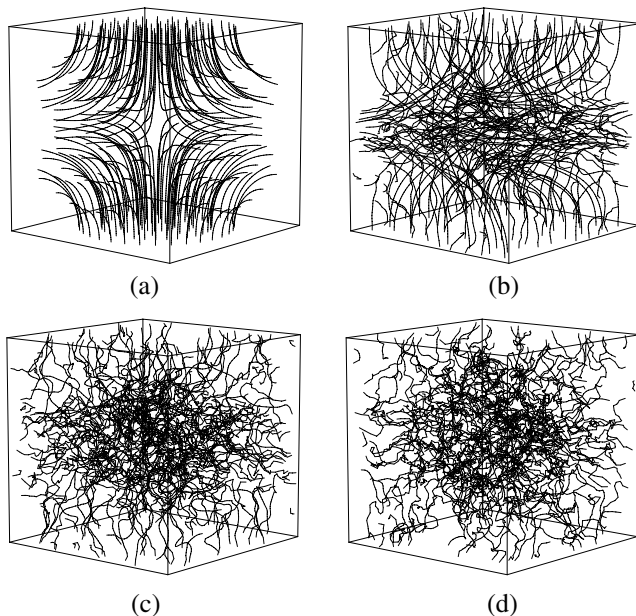


FIG. 1. Time evolution of the vortex tangle at $t = 0$ sec (a), $t = 30.0$ sec (b), $t = 50.0$ sec (c), and $t = 70.0$ sec (d).

comes from the velocity field near each vortex line [3,14], though the random vortices compose the turbulent velocity field. On the contrary, the spectrum for $k < 2\pi/l$ is strongly affected by the random vortex configuration. At $t = 0$ sec, the spectrum has a large peak at the smallest wave number, being flat in the intermediate range because there are only large vortices and no short-scale structure on them. Figure 2 shows that as the vortices become the homogeneous and isotropic vortex tangle the slope for $k < 2\pi/l$ converges to the Kolmogorov form $k^{-5/3}$.

The Kolmogorov law can be derived from the argument based on the picture of the cascade process [15]. In the inertial range the kinetic energy is transferred steadily from small k to large k without dissipation, and dissipated at the end of the inertial range. Thus, for the steady state, the energy dissipation rate $\epsilon = -dE/dt$ can be identified with the energy flux in the inertial range. Then the energy spectrum depends only on the wave number k and the energy dissipation rate $\epsilon = -dE/dt$, which leads to the Kolmogorov spectrum $E(k) = C\epsilon^{2/3}k^{-5/3}$. Here C is the (dimensionless) Kolmogorov constant of order unity.

In our previous papers, the cascade process without the mutual friction in superfluid turbulence was discussed [10]. Through lots of reconnections, the vortex tangle breaks up to smaller ones and this process proceeds self-similarly, and in our calculation the smallest vortex whose size is the order of the numerical space resolution $\Delta\xi$ is eliminated by the cutoff procedure. This resolution in our calculation is much larger than the dissipative scale in a real system. However, this numerical cutoff can be justified, because the cascade process at a small scale proceeds much faster than that at a large scale. Actually we showed the decay rate of the density of vortices was almost independent of the cutoff scale $\Delta\xi$. Figure 3 shows the energy dissipation rate ϵ due to the cutoff procedure in the dynamics of Fig. 1. After 70 sec, the tangle becomes isotropic and homogeneous losing the memory

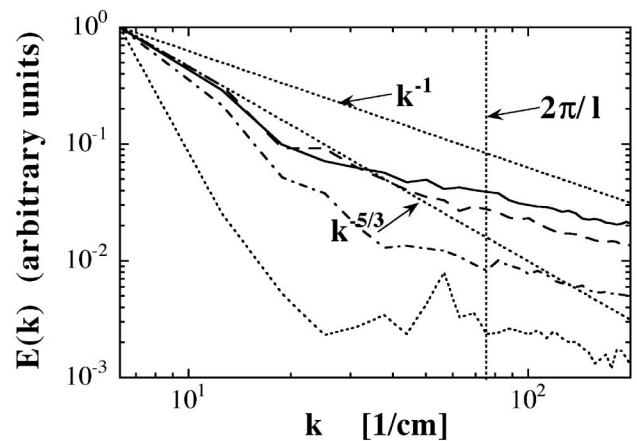


FIG. 2. The energy spectra of the vortex tangle at $t = 0$ sec (dashed line), $t = 30.0$ sec (dot-dashed line), $t = 50.0$ sec (long-dashed line), and $t = 70.0$ sec (solid line).

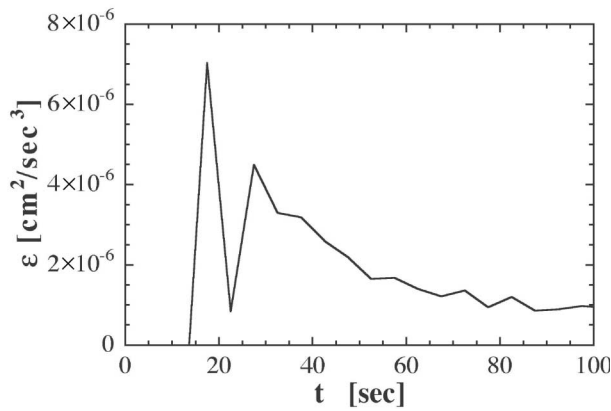


FIG. 3. The energy dissipation rate ϵ which was calculated from the dynamics of Fig. 1.

of the initial configuration, so the change of the dissipation rate becomes slow free from the artifact of the early large dissipation.

Next we compare quantitatively the energy spectrum at 70 sec with the Kolmogorov law $E(k) = C\epsilon^{2/3}k^{-5/3}$. The Kolmogorov constant C is known as the parameter of order unity. Here we use $C = 1$ and $\epsilon = 1.287 \times 10^{-6} \text{ cm}^2/\text{sec}^3$. Then we can determine uniquely the energy spectrum. Figure 4 shows the comparison between the energy spectrum at $t = 70$ sec and the Kolmogorov law with $C = 1$ and $\epsilon = 1.287 \times 10^{-6} \text{ cm}^2/\text{sec}^3$. The energy spectrum for $k < 2\pi/l$ is consistent with the Kolmogorov law not only on the wave number dependence but also on the absolute value. The dissipative mechanism due to the cutoff works only at the largest wave number $k \sim 2\pi/\Delta\xi = 343 \text{ cm}^{-1}$. However, the energy spectrum at the small k region is determined by that dissipation rate. This result supports just the picture of the inertial range. Although our spectrum has a k^{-1} region between the Kolmogorov region and the dissipative wave

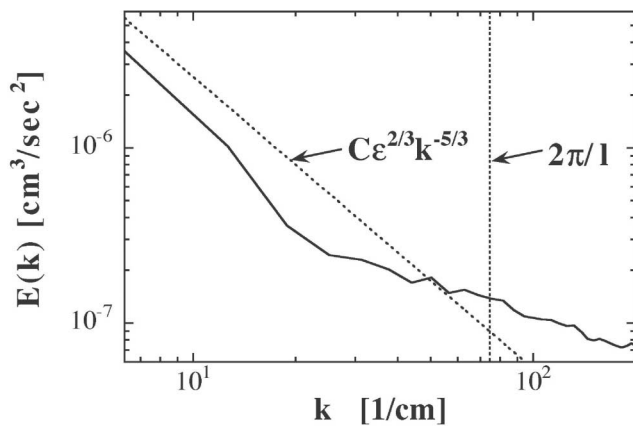


FIG. 4. Comparison of the energy spectrum (solid line) at $t = 70$ sec with the Kolmogorov law $E(k) = C\epsilon^{2/3}k^{-5/3}$ (dotted line) with $C = 1$ and $\epsilon = 1.287 \times 10^{-6} \text{ cm}^2/\text{sec}^3$.

number, the energy flux exists also in this region. The spectrum in this region includes the contribution coming from each vortex line and that of the energy flux, while the former is dominant [14]. The cascade process in this region will be discussed elsewhere.

The Kolmogorov law is the scaling property in k space and is closely related with the self-similarity of the turbulent velocity field in the real space. We devote the rest of this Letter to the following question: is this scaling property in k space related to the self-similarity of the vortex tangle in real space or not? However, in conventional turbulence it is very difficult to discuss this problem, because the viscous diffusion of vorticity makes the vortex configuration obscure. On the contrary, the characters of the superfluid turbulence are the definiteness of the vortex line due to the absence of the viscosity and the fixed circulation by the quantum effect. These characters allow us to describe the system by the topological configuration of a vortex tangle. Hence, in order to discuss the self-similarity in a real space, it is meaningful to investigate a vortex length distribution (VLD) $n(x)$, where $n(x)\Delta x$ represents the number of the vortices whose length is from x to $x + \Delta x$.

In order to suppress the fluctuation of the VLD, we use the time averaging procedure in the interval 4.0 sec. Figure 5 shows the averaged VLD at each time, where the largest scale in the x axis is the size of the cube and the smallest scale is the length of the numerical space resolution. At large scale the VLD is strongly affected by the effect of the boundary, and at small scale it decreases rapidly by the effect of the cutoff procedure. As the vortices approach the homogeneous and isotropic tangle, in the intermediate range the VLD comes to obey a scaling property $n(x) \propto x^\alpha$. Using the least squares fits, we determine this scaling exponent α at $t = 60.0$ sec in

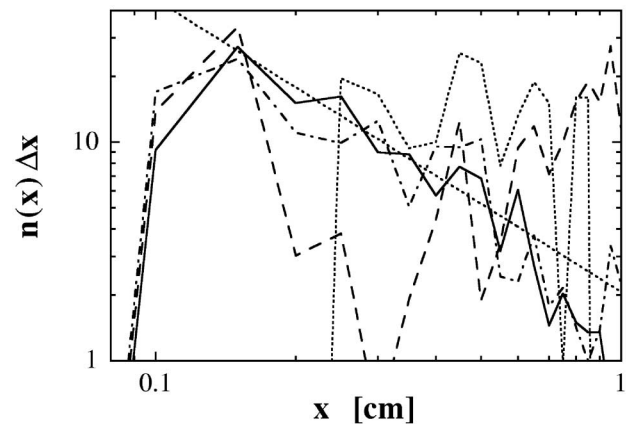


FIG. 5. The VLD $n(x)\Delta x$ at $t = 0$ sec (dashed line), $t = 20.0$ sec (long-dashed line), $t = 40.0$ sec (dot-dashed line), and $t = 60.0$ sec (solid line) for $\Delta x = 0.05$. The dashed line is determined by the least squares fits at $t = 60.0$ sec.

the region of $0.15 < x < 0.65$ (cm) and find $\alpha = -1.34 \pm 0.18$.

In conclusion, we have studied numerically the energy spectrum of the superfluid turbulence without the mutual friction. For $k > 2\pi/l$, the spectrum can be attributed to the contribution of each vortex line. For $k < 2\pi/l$, as the vortices approach the homogeneous and isotropic tangle, the slope of the spectrum converges to the Kolmogorov form $k^{-5/3}$. By using the energy dissipation rate due to the elimination of the smallest vortices, the spectrum for $k < 2\pi/l$ is consistent with the Kolmogorov law not only on the wave number dependence but also on the absolute value. The VLD $n(x)$ comes to obey a scaling property reflecting the self-similarity of the tangle.

We acknowledge W.F. Vinen for useful discussions. M.T. acknowledges support by a Grant-in-Aid for Scientific Research (Grant No. 12640357) by Japan Society for the Promotion of Science. S.N. thanks Russian Foundation of Basic Research (Grant No. 99-02-16942) for supporting that field.

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- [13] You can see the animation of this dynamics in <http://matter.sci.osaka-cu.ac.jp/bsr/taraki/movie.html>.
- [14] The energy spectrum of a straight vortex line can be derived. Since the velocity field of a straight vortex line is $\kappa/2\pi r$, the kinetic energy of a vortex line per unit mass is calculated as $E = (1/2) \int v^2(\mathbf{r}) d^3r = (\kappa^2 L/4\pi) \ln(D/a)$, where L is the total line length per unit volume and r is the distance from the vortex line. Since the energy spectrum $E(k) = Ck^\alpha$ can be defined as $E = \int_{2\pi/D}^{2\pi/a} E(k) dk$, the parameters α and C are determined as $\alpha = -1$ and $C = \kappa^2 L/4\pi$. For example, at $t = 70$ sec, $L = 151.9 \text{ cm}^{-2}$, so that $E(k)_{\text{line}} = (\kappa^2 L/4\pi) k^{-1} = 1.20 \times 10^{-7} \text{ cm}^3/\text{sec}^2$ for $k = 100 \text{ cm}^{-1}$. This is consistent with our numerical result $E(k) = 1.07 \times 10^{-7} \text{ cm}^3/\text{sec}^2$ for $k = 100 \text{ cm}^{-1}$ in Fig. 4.
- [15] See, for example, U. Frisch, *Turbulence* (Cambridge University Press, Cambridge, England, 1995).