Gaussian Transformations and Distillation of Entangled Gaussian States

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We prove that it is impossible to distill more entanglement from a single copy of a two-mode bipartite entangled Gaussian state via local Gaussian operations and classical communication. More generally, we show that any hypothetical distillation protocol for Gaussian states involving only Gaussian operations would be a deterministic protocol. Finally, we argue that the protocol considered by Eisert *et al.* [preceding Letter, Phys. Rev. Lett. **89**, 137903 (2002)] is the optimum Gaussian distillation protocol for two copies of entangled Gaussian states.

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Quantum entanglement is a key ingredient of many protocols for quantum information processing such as quantum teleportation [1]. Usually, the entangled particles are distributed among two distant parties traditionally called Alice and Bob. In practice, the transmission channel is always noisy and imperfect, which prevents Alice and Bob from sharing a maximally entangled state even if Alice can prepare such a state locally. Fortunately, the errors introduced by noisy quantum channels can be overcome by the so-called entanglement distillation protocols, by which Alice and Bob can extract from a large number of weakly entangled mixed states a smaller number of highly entangled almost pure states [2,3].

Recently, a great deal of attention has been devoted to quantum information processing with continuous quantum variables. Remarkably, linear optics, parametric amplifiers, and homodyne detectors suffice for implementation of many protocols including continuous-variable teleportation [4], cryptography [5], and cloning [6]. However, one important protocol missing in our toolbox is a feasible distillation protocol for continuous variables. We are particularly interested in distillation protocols for entangled Gaussian states because these states can easily be generated in the laboratory. Note also that necessary and sufficient conditions for entanglement of bipartite Gaussian states have been found [7–9]. The distillation protocols proposed so far involve rather complicated nonlinear transformations such as subtraction of a single photon [10] or a quantum nondemolition measurement of the total photon number in several modes [11]. It would be of great help to have a distillation protocol for continuous variables that could be implemented with linear optics and that would distill Gaussian entangled states. However, no such protocol is currently known and it is an open question whether such a distillation protocol exists at all.

In the present Letter, we attempt to shed some light on this issue by making use of the formalism of Gaussian completely positive (CP) maps [12,13]. These maps represent all transformations that can be carried out with the help of passive and active linear optical elements, homodyne detectors, and auxiliary optical modes prepared initially in Gaussian states. These transformations may be deterministic or probabilistic. In the latter case, we accept or reject the output state in dependence on the output of a quantum measurement (with some Gaussian probability distribution). We consider an arbitrary bipartite probabilistic Gaussian operation which can be implemented with the help of local operations and classical communication (LOCC). We prove that for input bipartite Gaussian states it is always possible to construct a deterministic LOCC Gaussian transformation that yields the same output state (for a fixed input) as a given probabilistic LOCC Gaussian transformation. This implies that it is impossible to distill more entanglement from a single copy of an entangled Gaussian state by means of Gaussian operations. This should be contrasted with distillation protocols for a single copy of a two-qubit entangled state where the LOCC operations may in some cases allow one to extract more entanglement [14,15]. In particular, any pure entangled two-qubit state can be transformed with certain probability via LOCC operations onto a maximally entangled Bell state. Furthermore, our results imply that any hypothetical Gaussian distillation protocol optimized for given shared entangled Gaussian states would be a deterministic protocol and we find a generic structure of this optimum protocol. A version of this optimum protocol where Alice and Bob share two identical copies of a Gaussian state with a symmetric covariance matrix has been considered by Eisert *et al.* [16] who proved that it is impossible to distill entanglement via this protocol. These findings thus strongly support the conjecture that it is impossible to distill the entangled Gaussian states via Gaussian operations.

We extensively exploit the isomorphism [17] between CP maps $\mathcal{M}$ and positive-semidefinite operators $\chi \geq 0$ on the tensor product of input and output Hilbert spaces $\mathcal{H} \otimes \mathcal{K}$. In terms of χ, the relation between input and output density matrices can be written as a partial trace over the input space,

$$\rho_{\rm out} = {\rm Tr}_{\rm in}[\chi \rho_{\rm in}^T \otimes \mathbb{1}_{\rm out}], \quad (1)$$

where T stands for the transposition in some fixed basis and $\mathbb{1}_{\rm out}$ denotes an identity operator on the output space. The operator χ can be obtained from a maximally entangled state on $\mathcal{H}^{\otimes 2}$, $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_{j=1}^{d}|j\rangle_1|j\rangle_2$, ($d = \dim\mathcal{H}$) if the CP map $\mathcal{M}$ is applied to one part of this state,

$$\chi = \mathcal{I}_1 \otimes \mathcal{M}_2[d\,|\psi\rangle\langle\psi|]. \qquad (2)$$

Here $\mathcal{I}$ stands for the identity transformation. The subscripts 1 and 2 and the tensor product indicate that the map $\mathcal{M}$ is applied to subsystem 2 of the entangled state $|\psi\rangle$ while nothing happens with subsystem 1.

In continuous-variable systems, we deal with infinite dimensional Hilbert spaces, and the maximally entangled state $|\psi\rangle$ becomes a tensor product of $N_{\rm in}$ (unphysical) two-mode infinitely squeezed vacuum states, where $N_{\rm in}$ is the number of input modes. Gaussian CP maps are defined as maps that transform Gaussian states onto Gaussian states. Gaussian CP maps are thus isomorphic to bipartite Gaussian quantum states χ. Now any Gaussian state χ is completely characterized by the first and second moments: mean values of quadratures and a covariance matrix $\mathbf{\Gamma}$. Define vector of quadratures $\boldsymbol{r} = (x_1, p_1, \ldots, x_N, p_N)^T$ where N is the total number of input + output modes. The elements of matrix Γ are defined as $\Gamma_{ij} = \langle \Delta r_i \Delta r_j\rangle + \langle \Delta r_j \Delta r_i\rangle$, where $\Delta r_i = r_i - \langle r_i\rangle$. Nonzero mean values of the quadratures of χ indicate that the map χ involves certain displacements. However, these operations can be performed locally and are therefore irrelevant for the entanglement properties and can be omitted. Thus, we can assume that $\langle r_i\rangle = 0$ and the CP map χ is fully described by the covariance matrix $\mathbf{\Gamma}$. It is convenient to split the matrix $\mathbf{\Gamma}$ into input and output parts and write

$$\mathbf{\Gamma} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{C}^T & \boldsymbol{B} \end{pmatrix}, \qquad (3)$$

where $\boldsymbol{A}$ stands for the covariance matrix of the "input" modes, $\boldsymbol{B}$ is the covariance matrix of the "output" modes, and $\boldsymbol{C}$ contains the input-output correlations. The map (1) rewritten in terms of the Wigner functions reads

$$W_{\rm out}(\boldsymbol{r}_{\rm out}) = (2\pi)^{N_{\rm in}} \int_{-\infty}^{\infty} W_\chi(\boldsymbol{r}_{\rm in}, \boldsymbol{r}_{\rm out}) W_{\rm in}(\boldsymbol{R}\boldsymbol{r}_{\rm in}) d\boldsymbol{r}_{\rm in}, \quad (4)$$

where $\boldsymbol{R} = \mathrm{diag}(1, -1, 1, -1, \ldots, 1, -1)$ is a diagonal matrix that represents the transposition in phase space ($x_j \to x_j$, $p_j \to -p_j$). It is convenient to deal with characteristic functions which are Fourier transforms of the Wigner functions, $C(\boldsymbol{\xi}) = \int_{-\infty}^{\infty} W(\boldsymbol{r}) \exp(i\boldsymbol{r}\cdot\boldsymbol{\xi}) d\boldsymbol{r}$. On expressing all Wigner functions in terms of the characteristic functions, we obtain from Eq. (4),

$$C(\boldsymbol{\xi}_{\rm out}) = (2\pi)^{-N_{\rm in}} \int_{-\infty}^{\infty} C_\chi(\boldsymbol{\xi}_{\rm in}, \boldsymbol{\xi}_{\rm out}) C_{\rm in}(-\boldsymbol{R}\boldsymbol{\xi}_{\rm in}) d\boldsymbol{\xi}_{\rm in}. \qquad (5)$$

Assuming an input Gaussian state $\rho_{\rm in}$ with covariance matrix $\mathbf{\Gamma}_{\rm in}$, $C_{\rm in}(\boldsymbol{\xi}) = \exp(-\frac{1}{4}\boldsymbol{\xi}^T\mathbf{\Gamma}_{\rm in}\boldsymbol{\xi})$, we find that the output state is also Gaussian with a covariance matrix

$$\mathbf{\Gamma}_{\rm out} = \boldsymbol{B} - \boldsymbol{C}^T(\boldsymbol{A} + \boldsymbol{R}\mathbf{\Gamma}_{\rm in}\boldsymbol{R})^{-1}\boldsymbol{C}. \qquad (6)$$

We now prove a very important feature of Gaussian CP maps. It holds that, for every input Gaussian state and a probabilistic (trace-decreasing) LOCC Gaussian CP map, there exists a deterministic (trace-preserving) LOCC Gaussian CP map that transforms the input state into the output state with the covariance matrix (6). The explicit construction of the trace-preserving map is inspired by recent works on the possibility of storing quantum dynamics in quantum states [18–20]. The trick is to encode the transformation into a bipartite state χ which then serves as a quantum channel in the teleportation [1,4]. In this way, the desired transformation is carried out with certain probability depending on the dimension of the Hilbert space. The continuous-variable analog of this scheme goes as follows. We prepare a Gaussian state χ with covariance matrix $\mathbf{\Gamma}$ given by Eq. (3) and carry out a Bell measurement on the input modes of the state χ and the input state $\rho_{\rm in}$. This measurement is performed separately for each corresponding pair of modes and consists of measuring the difference of the x quadratures and the sum of the p quadratures by means of balanced homodyne detectors (BHD) [4]. Let the vector $\boldsymbol{r}_d$ contain the outcomes of these measurements. The (non-normalized) Wigner function of the output modes conditioned on the outcome $\boldsymbol{r}_d$ reads

$$W_{\rm out}(\boldsymbol{r}_{\rm out} \mid \boldsymbol{r}_d) = \int_{-\infty}^{\infty} W_\chi(\boldsymbol{r}_{\rm in} + \boldsymbol{r}_d, \boldsymbol{r}_{\rm out}) W_{\rm in}(\boldsymbol{R}\boldsymbol{r}_{\rm in}) d\boldsymbol{r}_{\rm in}.$$

For an input Gaussian state, the covariance matrix of the output state is given by Eq. (6) and does not depend on $\boldsymbol{r}_d$. However, the output state is displaced by

$$\boldsymbol{r}_{\rm cond} = \boldsymbol{C}^T(\boldsymbol{A} + \boldsymbol{R}\mathbf{\Gamma}_{\rm in}\boldsymbol{R})^{-1}\boldsymbol{r}_d. \qquad (7)$$

If we know the input state and the transformation, then we can calculate $\boldsymbol{r}_{\rm cond}$ for given detected quadratures $\boldsymbol{r}_d$ and by means of suitable displacement transformation applied to the output state we can always set the coherent signal in the output state to zero. In this way we obtain in a deterministic manner an output state which has the covariance matrix (6). We should note here that many Gaussian CP maps, in particular, all trace-preserving maps, are represented by unphysical (infinitely squeezed) states χ. However, we can approximate such an unphysical state by a physical finitely squeezed state with an arbitrarily high accuracy and thus also approximate the transformation (6) with arbitrarily high precision.

Let us now turn our attention to the LOCC Gaussian CP maps. Obviously, every LOCC Gaussian map $\mathcal{M}_{\rm LOCC}$ is isomorphic to a Gaussian state χ which is separable with respect to Alice and Bob. [Note that according to Eq. (2)

Alice and Bob can prepare the state χ in their labs via LOCC operations.] A scheme for deterministic implementation of any LOCC Gaussian CP map (for a known input Gaussian state) is shown in Fig. 1 for the simplest case when there is a single input and a single output mode on each side. By means of LOCC operations, Alice and Bob prepare the four-mode state χ representing the CP map ($A1$ and $B1$ are input modes and $A2$ and $B2$ are output modes). Alice mixes her part of the input state in mode A_{in} with $A1$ on a balanced beam splitter and measures $x_{A1} - x_{A\text{in}}$ and $p_{A\text{in}} + p_{A1}$. Bob performs the same operations with his modes $B1$ and B_{in}. Alice and Bob exchange the results of their measurements via a classical communication channel and appropriately locally displace the modes $A2$ and $B2$, thereby producing deterministically the desired two-mode output state in modes A_{out} and B_{out}. More generally, it holds that every probabilistic Gaussian CP map can be replaced by its deterministic counterpart if the input is a known Gaussian state. Note also that the above protocol works only for Gaussian states. If the input state is not Gaussian, then it may happen that some trace-decreasing LOCC Gaussian CP maps will yield outputs that cannot be obtained with any trace-preserving LOCC Gaussian CP map.

A very important implication concerning distillation protocols is that we cannot distill more entanglement from a single copy of a two-mode bipartite entangled Gaussian state by means of LOCC Gaussian operations. This follows from the fact that any probabilistic LOCC Gaussian operation can be replaced by a deterministic one which yields the same output (for a given input Gaussian state). However, any reasonable measure of entanglement must be nonincreasing under deterministic LOCC operations. For instance, it is impossible to extract more entanglement from a single copy of a two-mode Gaussian state via entanglement swapping based on the Braunstein-Kimble teleportation scheme [21].

Let us have a more detailed look at the structure of the bipartite LOCC Gaussian CP maps. It was shown by Werner and Wolf [22] that a bipartite Gaussian state is separable if and only if it can be transformed via local symplectic transformations [23] onto a state with positive Glauber-Sudarshan representation, i.e., a state which is a convex mixture of coherent states and is not squeezed. We can thus write

$$\chi_{\text{LOCC}} = \int_{-\infty}^{\infty} P(\boldsymbol{\alpha}, \boldsymbol{\beta}) S_A |\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}| S_A^{\dagger} \otimes S_B |\boldsymbol{\beta}\rangle\langle\boldsymbol{\beta}| S_B^{\dagger} d\boldsymbol{\alpha} d\boldsymbol{\beta},$$

where $P(\boldsymbol{\alpha}, \boldsymbol{\beta}) \geq 0$ is a positive-semidefinite Gaussian function, $|\boldsymbol{\alpha}\rangle$ and $|\boldsymbol{\beta}\rangle$ denote (multimode) coherent states of Alice's and Bob's modes, and S_A and S_B denote symplectic transformations. We have seen that it is impossible to distill more entanglement from a single copy of a two-mode Gaussian state by means of LOCC Gaussian transformations. What if Alice and Bob possess several copies? Assume that they apply the LOCC Gaussian map χ_{LOCC} to their states. This distillation map takes all copies as an input and yields a single copy of a two-mode state shared by Alice and Bob. This output state is a mixture of states with identical covariance matrices and varying displacements. In terms of Wigner functions, we can write

$$W_{\text{out}}(\boldsymbol{r}) = \int_{-\infty}^{\infty} P(\boldsymbol{\alpha}, \boldsymbol{\beta}) W(\boldsymbol{r} - \boldsymbol{r}_d(\boldsymbol{\alpha}, \boldsymbol{\beta})) d\boldsymbol{\alpha} d\boldsymbol{\beta}, \quad (8)$$

where $\boldsymbol{r}_d(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is the displacement. However, all states with Wigner functions $W(\boldsymbol{r} - \boldsymbol{r}_d)$ exhibit the same entanglement, because entanglement depends only on the covariance matrix and not on the displacement. Hence, it is always optimal to choose an LOCC CP map which is represented by a *pure* Gaussian state. An analogous situation arises in distillation of a single pair of entangled qubits where it is optimum to apply local filtering operation [15] (this is a trace-decreasing CP map whose Kraus decomposition contains only one term and the CP map is thus represented by a pure state).

Since we assume that we know the state that we want to distill, we can transform any LOCC trace-decreasing map onto a trace-preserving map; hence, the optimum protocol will be deterministic. Consider now the simplest case when Alice and Bob share two pairs of entangled Gaussian states. The transformation on Alice's side is represented by a pure three-mode Gaussian state, which can be obtained from three-mode vacuum via some three-mode symplectic transformation. This three-mode state splits into two input modes and one output mode. By means of "local" symplectic transformations on input and output modes, we can transform the covariance matrix of this state into the form

$$\boldsymbol{\Gamma} = \begin{pmatrix} a & 0 & 0 & 0 & d_1 & 0 \\ 0 & a & 0 & 0 & d_3 & d_2 \\ 0 & 0 & b & 0 & e_1 & 0 \\ 0 & 0 & 0 & b & 0 & e_2 \\ d_1 & d_3 & e_1 & 0 & c & 0 \\ 0 & d_2 & 0 & e_2 & 0 & c \end{pmatrix}. \quad (9)$$

The reduced density matrix of each mode is a density matrix of the thermal state and there are no correlations

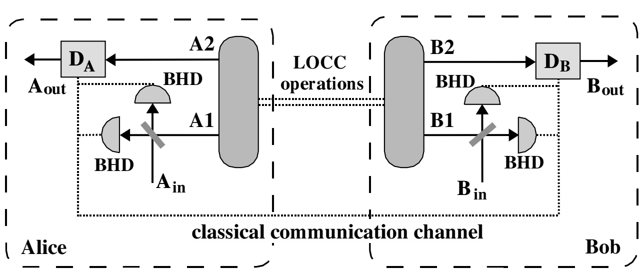

FIG. 1. Setup for implementation of a deterministic LOCC Gaussian CP map that is for a single fixed input Gaussian state equivalent to a given probabilistic LOCC Gaussian CP map.

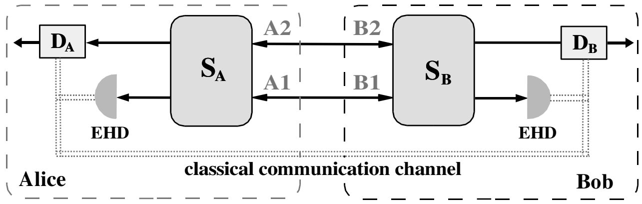

FIG. 2. Optimal distillation protocol for two copies of two-mode entangled Gaussian states.

between the two input modes. Since the whole three-mode state is pure, the density matrix of the two input modes has the same eigenvalues as the reduced density matrix of the output mode. This is possible only if one of the input modes is in a pure vacuum state. This leads to $a = 1$, $b = c$, $d_1 = d_2 = d_3 = 0$. We conclude that we can prepare any pure three-mode Gaussian state from a two-mode squeezed vacuum state and a vacuum state if we apply single-mode symplectic transformation $S_{\rm out}$ to the output mode and a two-mode symplectic transformation $S_{\rm in}$ to the two input modes. We may thus write the transformation on Alice's side in the form

$$\chi_A = S_{\rm in} \otimes \mathbb{1}_{\rm out}(|0\rangle\langle 0| \otimes \chi_{A0})S_{\rm in}^{\dagger} \otimes \mathbb{1}_{\rm out}, \qquad (10)$$

where χ_{A0} is an operator on Hilbert space of two modes (one input and one output). To see what are the implications, we insert this expression into formula (1) and for the moment do not consider Bob's states. We get

$$\rho_{\rm out} = {\rm Tr}_{\rm in}[|0\rangle\langle 0| \otimes \chi_{A0}(S_{\rm in}^{T}\rho_{\rm in}S_{\rm in}^{*})^{T} \otimes \mathbb{1}_{\rm out}]. \qquad (11)$$

The transformation thus reduces to the following three steps: (i) Apply symplectic transformation $S_{\rm in}^{T}$ to the input two-mode state. (ii) Project the first mode onto a vacuum state. (iii) Apply a CP map χ_{A0} to the second mode. The transformation on Bob's side has the same structure. This protocol can be further simplified. After Alice and Bob project one of the modes onto a vacuum state, they possess only a single mode each. The application of the local maps χ_{A0} (χ_{B0}) cannot increase entanglement, because we have shown that it is impossible to distill a single copy of a two-mode entangled Gaussian state by means of LOCC Gaussian operations. Thus we need not consider the transformations χ_{A0} (χ_{B0}), and the resulting optimal simplified distillation scheme is shown in Fig. 2. Both Alice and Bob locally apply some two-mode symplectic transformations S_A and S_B to their modes $A1$, $A2$ and $B1$, $B2$, respectively. Subsequently, they both project the modes $A1$ and $B1$ into coherent states $|\alpha\rangle$ and $|\beta\rangle$ with the help of eight-port homodyne detectors (EHD). Finally, they exchange the results of their measurements and displace appropriately the output states. This scheme is deterministic and represents the optimal Gaussian distillation protocol for Gaussian states. Eisert *et al.* considered a special kind of this protocol where the eight-port homodyne detectors are replaced with balanced homodyne detectors and they proved that it is impossible to distill entanglement from two identical copies of a two-mode Gaussian state with symmetrical covariance matrix via this protocol [16]. All these results strongly support the conjecture that it is impossible to distill entangled Gaussian states with Gaussian operations.

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Note added.—After this work was completed, I learned that Giedke and Cirac [24] have also investigated the properties of trace-decreasing Gaussian CP maps and they independently obtained similar results. Moreover, they proved that the distillation of Gaussian states with Gaussian operations is impossible for an arbitrary number of modes per site.

[1] C. H. Bennett *et al.*, Phys. Rev. Lett. **70**, 1895 (1993).
[2] C. H. Bennett *et al.*, Phys. Rev. Lett. **76**, 722 (1996).
[3] D. Deutsch *et al.*, Phys. Rev. Lett. **77**, 2818 (1996).
[4] S. L. Braunstein and H. J. Kimble, Phys. Rev. Lett. **80**, 869 (1998); A. Furusawa *et al.*, Science **282**, 706 (1998).
[5] T. C. Ralph, Phys. Rev. A **61**, 010303(R) (2000); F. Grosshans and P. Grangier, Phys. Rev. Lett. **88**, 057902 (2002); Ch. Silberhorn *et al.*, *ibid.* **88**, 167902 (2002).
[6] S. L. Braunstein *et al.*, Phys. Rev. Lett. **86**, 4938 (2001); J. Fiurášek, Phys. Rev. Lett. **86**, 4942 (2001).
[7] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000).
[8] L. M. Duan *et al.*, Phys. Rev. Lett. **84**, 2722 (2000).
[9] G. Giedke *et al.*, Phys. Rev. Lett. **87**, 167904 (2001).
[10] T. Opatrný *et al.*, Phys. Rev. A **61**, 032302 (2000).
[11] L. M. Duan *et al.*, Phys. Rev. Lett. **84**, 4002 (2000).
[12] G. Lindblad, J. Phys. A **33**, 5059 (2000).
[13] J. Eisert and M. B. Plenio, Phys. Rev. Lett. **89**, 097901 (2002).
[14] C. H. Bennett *et al.*, Phys. Rev. A **53**, 2046 (1996).
[15] A. Kent *et al.*, Phys. Rev. Lett. **83**, 2656 (1999).
[16] J. Eisert, S. Scheel, and M. B. Plenio, preceding Letter, Phys. Rev. Lett. **89**, 137903 (2002).
[17] A. Jamiolkowski, Rep. Math. Phys. **3**, 275 (1972).
[18] M. A. Nielsen and I. L. Chuang, Phys. Rev. Lett. **79**, 321 (1997).
[19] W. Dür and J. I. Cirac, Phys. Rev. A **64**, 012317 (2001).
[20] M. Hillery *et al.*, Phys. Rev. A **65**, 022301 (2002).
[21] S. Parker *et al.*, Phys. Rev. A **61**, 032305 (2000).
[22] R. F. Werner and M. M. Wolf, Phys. Rev. Lett. **86**, 3658 (2001).
[23] R. Simon, N. Mukunda, and B. Dutta, Phys. Rev. A **49**, 1567 (1994).
[24] G. Giedke and J. I. Cirac, quant-ph/0204085 [Phys. Rev. A (to be published)].