Some Generic Aspects of Bosonic Excitations in Disordered Systems

V. Gurarie and J. T. Chalker

Theoretical Physics, Oxford University, 1 Keble Road, Oxford OX1 3NP, United Kingdom

We consider noninteracting bosonic excitations in disordered systems, emphasizing generic features of quadratic Hamiltonians in the absence of Goldstone modes. We discuss relationships between such Hamiltonians and the symmetry classes established for fermionic systems. We examine the density $\rho(\omega)$ of excitation frequencies ω, showing how the universal behavior $\rho(\omega) \sim \omega^4$ for small ω can be obtained both from general arguments and by detailed calculations for one-dimensional models.

An understanding of universal properties of excitations in disordered systems occupies a central place in condensed matter physics. Much of the work in this area has focused on systems of noninteracting fermions, as models for quasiparticles in disordered conductors or superconductors with interactions treated at the mean-field level. Classification of such systems according to symmetry provides an important starting point, and in many instances the possibilities are the three represented by the Wigner-Dyson random matrix ensembles [1]. A recent development, however, has been the recognition that there exist symmetry classes additional to those of the Wigner-Dyson ensembles. These additional symmetry classes arise in fermionic systems which have either chiral symmetry, as for tight-binding models on bipartite lattices with only off-diagonal disorder [2], or particle-hole symmetry, as for the Bogoliubov–de Gennes Hamiltonian in disordered superconductors [3]. A characteristic feature of both cases is that eigenstates at positive and negative energies are related in pairs and zero energy emerges as a special point in the spectrum.

It is natural to anticipate that similar mathematical structure may be important for the theory of noninteracting bosonic excitations or classical harmonic modes in disordered systems. Our aim in the following is to examine how far this is the case and what consequences it has. Studies of systems of this type have a long history, with celebrated early work by Dyson [4] on the dynamics of a disordered chain of masses and springs, and applications which include phonons in disordered solids [5] and spin waves in random magnets [6]. An obvious parallel between excitations in these systems and those in fermionic systems belonging to the one of the additional symmetry classes is that bosonic excitations arise in positive and negative frequency pairs, and zero is a special point in the frequency spectrum. A second, more formal parallel is that random magnets, for example, in common with superconductors, may give rise to quadratic Hamiltonians containing terms that annihilate and create particle pairs. There are also clear differences between systems with bosonic and fermionic excitations, which prevent an elementary transcription of established ideas. Most impor-

tantly, while the exclusion principle guarantees that a fermionic system with a quadratic Hamiltonian has a ground state, for a corresponding bosonic system constraints must be imposed to ensure that it is stable. As a consequence, even to construct a phenomenological treatment such as random matrix theory for bosonic excitations, it is necessary to keep in mind their origins in an appropriate nonlinear problem. In turn, a distinction arises that is specific to bosonic excitations, between those which are Goldstone modes and those which are not. We focus below on the latter and discuss elsewhere systems with continuous symmetry [7].

Any quadratic bosonic Hamiltonian can be written in the forms

$$H = \sum_{i,j=1}^{N} [M_{ij}p_i p_j + K_{ij}x_i x_j + C_{ij}(x_i p_j + p_j x_i)]$$
$$\equiv \sum_{i,j=1}^{2N} \mathcal{H}_{ij}\xi_i \xi_j \equiv \frac{1}{2}(\mathbf{a}^\dagger \ \mathbf{a})\begin{pmatrix} \Gamma & \Delta \\ \Delta^\dagger & \Gamma^T \end{pmatrix}\begin{pmatrix} \mathbf{a} \\ \mathbf{a}^\dagger \end{pmatrix}. \quad (1)$$

Here x_i and p_i are the coordinates and momenta of the oscillators, $\xi_i = p_i$, $\xi_{N+i} = x_i$ with $1 \le i \le N$, and the vectors $\mathbf{a}^\dagger$, $\mathbf{a}$ have as entries bosonic creation and annihilation operators $a_i^\dagger, a_i = (x_i \pm ip_i)/\sqrt{2}$. The matrix $\mathcal{H}$ is real symmetric, Γ is Hermitian, and Δ is symmetric. The condition for time-reversal invariance is that $C = 0$ or, equivalently, that Γ and Δ are real.

We are concerned with spectral properties of models of this kind which are generic when $\mathcal{H}$ is random. One anticipates that these will be found at small excitation frequencies ω_i, and we concentrate particularly on their average density $\rho(\omega) = N^{-1}\langle \sum \delta(\omega - \omega_i) \rangle$ at small ω. Indeed, the final expression of Eq. (1) illustrates the parallel between bosonic models and the Bogoliubov–de Gennes Hamiltonian, from which universal behavior has been derived for the density of quasiparticle states in disordered, gapless superconductors [3]. In fermionic systems such behavior arises essentially from the interplay of disorder and symmetry, and does not depend on details of the distribution of Hamiltonian matrix elements. By contrast, for the bosonic systems we discuss, we show that

the requirements of stability impose features on the distribution of $\mathcal{H}$ which determine the form of $\rho(\omega)$ at small ω.

A simple argument leading to this conclusion has been given previously [8–10]. In outline it is as follows. Recall that the Hamiltonian of Eq. (1) is characterized not only by the oscillator frequencies ω_i but also by the eigenvalues κ_n of the matrix $\mathcal{H}$, which we refer to as oscillator *stiffnesses*; let $d(\kappa) = (2N)^{-1}\langle \sum \delta(\kappa - \kappa_n)\rangle$ be their average density. From a discussion of the curvature distribution at absolute minima of random functions of one variable, it is suggested [8] that $d(\kappa) \propto \kappa^{3/2}$ for small κ. Then, using the relation $\omega_i^2 = \kappa_1 \kappa_2$ which holds in a single-mode system, one arrives at $\rho(\omega) \propto d(\omega^2)\,\omega \propto \omega^4$ for small ω. A weakness at both steps in this argument is that 1 degree of freedom is treated in isolation. We show here how the same behavior emerges without such a restriction.

In a stable system, stiffnesses are positive and frequencies are real. It is helpful to introduce a description that guarantees this property. To this end, write $\mathcal{H}$ as a square of a real matrix Q, in the form $\mathcal{H} = Q^T Q$ (possible provided all $E_n \geq 0$). Also, note that frequencies are the eigenvalues of an auxiliary matrix $\mathcal{H}' = \sigma_2 \mathcal{H}$, where $\sigma_2 = \sigma_y \otimes I_N$, σ_y is the usual Pauli matrix, and I_N is the $N \times N$ identity matrix [11]. Since the eigenvalues of $\sigma_2 Q^T Q$ coincide with those of $\Omega = Q\sigma_2 Q^T$, which is Hermitian and antisymmetric, frequencies are real and occur in pairs $\pm\omega_i$. While there is in general no simple relation between stiffnesses and frequencies, several special cases represent important exceptions. If [in Eq. (1)] $M = I_N$ and $C = 0$ (as for vibrational problems with all masses equal), then $\omega_i = \pm\sqrt{\kappa_i}$, where the nontrivial stiffnesses appearing here are the eigenvalues of K; whereas if $M = K$ and $C = 0$ (as for a random-bond ferromagnet), then $\omega_i = \pm\kappa_i$.

Moving beyond these special cases, a straightforward approach to multimode problems comes from concentrating on disorder realizations in which one stiffness, say, κ_1, is much smaller than all others. The probability density for such disorder realizations varies as $\kappa_1^{3/2}$, by the arguments of Ref. [8]. In the limiting case $\kappa_1 = 0$, two eigenvalues of Ω vanish [since $\det(\mathcal{H}) = \det(\Omega)$, and since the eigenvalues of Ω are paired], while for $0 < \kappa_1 \ll \kappa_n$, $n \neq 1$, perturbation theory yields a pair of excitation frequencies $\omega_i \propto \pm\sqrt{\kappa_1}$. As a result, $\rho(\omega) \propto \omega^4$ for $\omega \ll \omega_p$, where ω_p is the lowest excitation frequency in a typical sample. More generally, consider a macroscopic system in which low frequency modes are localized with a localization length which remains finite as $\omega \to 0$. Treating each localization volume independently, we obtain $\rho(\omega) \propto \omega^4$ for $\omega \ll \omega_p$, where ω_p is in this case the lowest excitation frequency in a typical localization volume.

Turning to detailed calculations, it is useful to construct a formulation in which Q appears linearly by considering the enlarged matrices

$$\tilde{\mathcal{H}} = \begin{pmatrix} 0 & Q \\ Q^T & 0 \end{pmatrix} \quad \text{and} \quad \tilde{\mathcal{H}}' = \begin{pmatrix} 0 & Q \\ \sigma_2 Q^T & 0 \end{pmatrix}. \quad (2)$$

Clearly, the eigenvalues of $\mathcal{H}$ and $\mathcal{H}'$ are squares of those of $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{H}}'$, respectively. The 2×2 structure of $\tilde{\mathcal{H}}$ displayed in Eq. (2) is interesting partly because it is the defining feature of the chiral symmetry classes, studied previously in a variety of contexts [2,12–16], while matrices with the structure of $\tilde{\mathcal{H}}'$ constitute a new, chiral bosonic problem. Whereas much past work on chiral symmetry classes has taken the elements of Q to be independent random variables, drawn from a given distribution, it is clear following our introductory discussion that a central concern in our case is to determine the distribution of $\mathcal{H}$ and hence (up to left multiplication by an orthogonal matrix) that of Q. It does not seem obvious in advance whether Q can be chosen with a simple form (for example, with only short-range couplings) for any given problem. In fact, we have found convenient, explicit expressions for Q in several one-dimensional examples [7]. In addition, the approach proves useful even for problems in which Q is known only implicitly, as we now show.

We consider the random field XY model, which has been studied extensively as a description of pinned charge density waves [17]. It has the Hamiltonian

$$H = \int_0^L \left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_x \phi)^2 + h(\phi, x) \right] dx, \quad (3)$$

where Π is the momentum conjugate to the angle ϕ, $h(\phi, x) = h(x)\cos[\phi(x) - \chi(x)]$, and $h(x)$ and $\chi(x)$ are the random amplitude and phase of an applied field, with $\langle h(\phi, x)\rangle = 0$ and $\langle h(\phi, x)h(\phi', x')\rangle = \delta(x - x')h_0 \cos(\phi - \phi')$. This continuum problem has, in the discrete notation used above, $M = I_N$, $C = 0$, and hence $\omega = \pm\sqrt{\kappa_i}$. We are therefore concerned only with K: for the sake of an obvious analogy, we refer to it as a Hamiltonian, denoting it by $\mathcal{H}$ and its eigenvalues by E in place of κ.

Stationary configurations of the field ϕ satisfy

$$-\partial_x^2 \phi + \partial_\phi h(\phi, x) = 0, \quad (4)$$

while amplitudes ψ of normal mode excitations about the ground-state ϕ_0, with frequency $\omega = \pm\sqrt{E}$, obey

$$\mathcal{H}\psi \equiv -\partial_x^2 \psi + \partial_\phi^2 h(\phi_0, x)\psi = E\psi. \quad (5)$$

We want to find the density of states for the Hamiltonian of Eq. (5). In this form, this problem has been the subject of many publications [17–19]. Summarizing what is known, a characteristic energy scale, the pinning energy [20] E_p, separates two regimes. For $E \gg E_p$ the density of states is only weakly affected by the random field and can be computed perturbatively in the field strength. For $E \ll E_p$ the density of states is strongly influenced by disorder and believed to vary with E as a power law, $d(E) \propto E^\beta$, but the value of β remains controversial

[9,10,17]. Here we show, in argeement with arguments due to Aleiner and Ruzin [9] and to Fogler [10], that $\beta = 3/2$.

Since Eq. (5) has the form of a Schrödinger equation with a random potential $\partial_\phi^2 h(\phi_0, x)$, it is tempting to anticipate behavior familiar from simple choices of potential distribution, such as Gaussian white noise. That would be too naive, however, because—as for any bosonic Hamiltonian—the spectrum of $\mathcal{H}$ is positive and so the potential cannot be arbitrarily random. To investigate the implications of this fact, it is useful to follow Feigelman [18], introducing the notion of a partial energy, $\mathcal{E}(\phi, y)$: the ground-state energy of the half-chain with coordinate values $0 \leq x \leq y$ and boundary condition $\phi(y) = \phi$. Interpreting Eq. (4) as an equation of motion for a system with space coordinate ϕ and time coordinate x, $\mathcal{E}(\phi, y)$ plays the role of an action and satisfies the Hamilton-Jacobi equation

$$\partial_x \mathcal{E} + \frac{1}{2}(\partial_\phi \mathcal{E})^2 = h(\phi, x), \tag{6}$$

while the classical trajectory $\phi_0(x)$ obeys $d\phi_0/dx = \partial_\phi \mathcal{E}$. Now define $V(x) = \partial_\phi^2 \mathcal{E}(\phi_0(x), x)$, which we call the chiral potential. As a direct consequence of Eq. (6) it satisfies

$$\frac{d}{dx}V + V^2 = \partial_\phi^2 h(\phi_0(x), x). \tag{7}$$

The Hamiltonian of Eq. (5) can therefore be rewritten as $\mathcal{H} = Q^T Q$, where

$$Q = -\frac{d}{dx} + V(x). \tag{8}$$

Thus the bosonic problem of Eq. (5) is equivalent to a one-dimensional chiral problem specified by Eqs. (2) and (8), in which $V(x)$ should be determined along with $\phi_0(x)$.

At this point we can draw on the work of Comtet *et al.* [13], in which one-dimensional chiral problems of this kind have been analyzed in detail. In particular, they show that if the chiral potential $V(x)$ has a positive average $\langle V(x) \rangle = E_p^{1/2}$ (sometimes referred to as staggering [16]), then the low-lying ($E \ll E_p$) states of Eq. (2) are localized with a localization length $\xi \sim E_p^{-1/2}$. In this regime, the integrated density of states $N(E) = \int_0^E d(E')dE'$ can be calculated as the probability of a negative fluctuation of $V(x)$ in the interval $x_1 < x < x_2$ for which $2\int_{x_1}^{x_2} dxV(x) \equiv 2U < \log(E)$. The probability for a rare event of this kind is expected under many circumstances to vary with E as $\exp[\alpha \log(E)] = E^\alpha$, but the value of α depends on the details of the distribution of $V(x)$. In the following, we establish the equivalence implied by our notation, between $\langle V(x) \rangle^2$ and pinning energy, and calculate the low-lying density of states for the random field XY model by studying rare negative fluctuations of the particular chiral potential that arises in this context.

To appreciate why $\langle V(x) \rangle > 0$, consider the ground state of the half-chain as a function of the boundary condition, ϕ: it varies smoothly except for jumps at a small number of isolated points within one period. At these points $\mathcal{E}(\phi, y)$ is continuous but has an upward cusp [10]; since $\mathcal{E}(\phi, y)$ is periodic, an average of $V(x) \equiv \partial_\phi^2 \mathcal{E}$ computed excluding these cusps (as on ϕ_0) is naturally positive, and its size can be estimated from the average sum of discontinuities in $\partial_\phi \mathcal{E}$. Alternatively, one can attempt a more detailed analysis of Eq. (6), which is similar to the Kardar-Parisi-Zhang equation [21], but with two differences. One, the absence of a diffusion term $D\partial_x^2 \mathcal{E}$, is unimportant since it is well known that the limit $D \to 0$ in the KPZ equation yields solutions which are the global minimum of Eq. (3). The second, involving the nature of noise correlations, is more significant: while studies of the KPZ equation deal with noise that is uncorrelated in both x and ϕ, we are concerned with noise that has long-range correlations in ϕ. This situation is familiar in the context of Burgers turbulence [22], and indeed, introducing $u = \partial_\phi \mathcal{E}$ one finds the Burgers equation $u_x + uu_\phi = \partial_\phi h$. Cusps in $\mathcal{E}(\phi, y)$ as a function of ϕ correspond to the shock waves of Burgers turbulence, and using dimensional analysis the dependence on noise strength $\langle V(x) \rangle \sim h_0^{1/3}$ can be obtained from Eq. (6), as in the standard picture of the pinning energy.

Turning to fluctuations in $V(x)$, a direct attack, via a solution of Eq. (6), is not appropriate since many aspects of Burgers turbulence remain controversial. Instead, we treat Eq. (7) as a Langevin equation for $V(x)$ with $\partial_\phi^2 h(\phi_0(x), x)$ playing the role of a random force. If $\phi_0(x)$ is determined by minimizing Eq. (3) for fixed $\phi(0)$ and $\phi(L)$, the resulting random force has built-in correlations which ensure that $V(x)$ remains bounded. Alternatively, one may solve Eq. (4) for $\phi(x)$ with fixed initial conditions, $\phi(0)$ and $\partial_x \phi(0)$. In this case, the random force is uncorrelated and appropriate values for $\partial_x \phi(0)$ generate ground states corresponding to different choices of $\phi(L)$. Other values of $\partial_x \phi(0)$, however, correspond to maxima of Eq. (3), and still others to higher energy local minima. From the perspective of the Langevin equation, a maximum of Eq. (3) is signalled by escape of $V(x)$ towards large negative values. Those noise realizations that result in escape should be eliminated by supplementing Eq. (7) with an absorbing boundary condition at $V(x) = -\infty$. The trajectories of $V(x)$ that are not absorbed should be weighted in two ways. First, we must eliminate solutions to Eq. (4) which are only local minima of Eq. (3). They are separated from the absolute minimum by maxima and hence have neighboring trajectories on which $V(x)$ escapes. The necessary weight is therefore the probability that no near neighbors of a given trajectory are absorbed. Second, we weight trajectories representing the absolute minimum for different $\partial_x \phi(0)$ in such a way as to give a uniform density on $\phi(L)$.

Omitting in the first instance both these weights, the form of the probability distribution $P(U)$ at large negative U can be found by passing from the Langevin equation to a path integral, fixing U with a Lagrange multiplier, and calculating the lowest eigenvalue of the corresponding Fokker-Planck operator, which is nonzero because of the absorbing boundary condition. In this way we obtain $P(U) \sim e^{2U}$ as $U \to -\infty$. To calculate the weighting factors, we consider a family of trajectories parametrized by s, with coordinate $\phi(x) + \eta(s, x)$ and chiral potential $V(x) + W(s, x)$, where $\eta(s, x)$ and $W(s, x)$ are assumed small. Deriving linearized evolution equations by expanding Eqs. (4) and (7), and solving these, we find $\eta(s, x_2) = e^U \eta(s, x_1)$ and $W(s, x_2) = e^{-2U} F(s)$, where

$$F(s) = W(s, x_1)$$
$$+ \eta(s, x_1) \int_{x_1}^{x_2} \exp\left(3 \int_{x_1}^{x} V(y)dy\right) \partial_\phi^3 h(\phi, x)dx. \quad (9)$$

In order that no trajectories neighboring $\phi(x)$ are absorbed, we require $W(s, x_2) \gtrsim -1$ for all s and hence, at large negative U, $F(s) \gtrsim -e^{2U}$, which occurs with probability e^U. Of the surviving trajectories, we should retain only those that have a fluctuation in the integrated chiral potential similar to U. These satisfy the condition $W(s, x_2) \lesssim 1$, and hence $F(s) \lesssim e^{2U}$, implying $|\eta(s, x_1)| \lesssim e^U$ and $|\eta(s, x_2)| \lesssim e^{2U}$, thus generating a weight e^{2U}. Combining these factors, the probability density that U is large and negative on $\phi_0(x)$ varies as e^{5U}, yielding $N(E) \sim E^{5/2}$, and hence $\rho(\omega) \sim \omega^4$, as anticipated above. A detailed account of these arguments will be presented elsewhere [7].

The mapping described here of Eq. (3) into a chiral problem also goes through for a discrete version of the XY spin chain. Figure 1 shows the result of a computation of $N(E)$ for a XY spin chain of 10^6 sites, with Hamiltonian $H = \sum_i \{h_i(\phi_i) - \cos(\phi_i - \phi_{i-1})\}$, where $h_i(\phi) = A_i \cos(\phi - \chi_i)$, and $[A \cos(\chi), A \sin(\chi)]$ is uniformly distributed on a disc of radius 0.01. Results are consistent with the power laws expected for both $E \ll E_p$ and $E \gg E_p$.

We note finally that, while the random field XY chain represents a special case in the sense noted above, since frequencies and stiffnesses are simply related, we expect the same form more generally for $\rho(\omega)$, as implied by the discussion preceding Eq. (2). Calculations for the random field Heisenberg spin chain lead to two-channel chiral problems involving $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{H}}'$. For such problems we obtain [7] $d(\kappa) \sim \kappa^{3/2}$ and $\rho(\omega) \sim \omega^4$.

The experimental situation is not clear-cut. Frequency dependent conductivity measurements on pinned charge density waves, summarized by Fogler [10], do not provide evidence for $\rho(\omega) \propto \omega^4$, perhaps because they are restricted to relatively high temperatures. By contrast, such an excitation spectrum, superimposed on a phonon
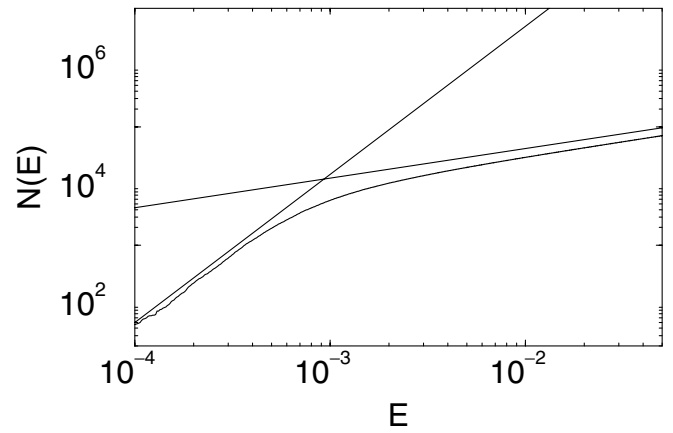

FIG. 1. The integrated density of stiffnesses, compared with the theoretically predicted powers, $\beta + 1 = 5/2$ for $E \ll E_p$ and $\beta + 1 = 1/2$ for $E \gg E_p$.

background, is deduced from specific-heat and neutron-scattering data for glasses [23].

[1] For a review, see M. L. Mehta, *Random Matrices* (Academic Press, San Diego, 1991).
[2] R. Gade, Nucl. Phys. **B398**, 499 (1993).
[3] A. Altland and M. R. Zirnbauer, Phys. Rev. B **55**, 1142 (1997).
[4] F. Dyson, Phys. Rev. **92**, 1331 (1953).
[5] Y. M. Galperin et al., Adv. Phys. **38**, 669 (1989).
[6] See, e.g., L. R. Walker and R. E. Walstedt, Phys. Rev. Lett. **38**, 514 (1977).
[7] V. Gurarie and J. T. Chalker (unpublished).
[8] M. A. Il'in et al., Sov. Phys. JETP **65**, 165 (1987).
[9] I. L. Aleiner and I. M. Ruzin, Phys. Rev. Lett. **72**, 1056 (1994).
[10] M. M. Fogler, Phys. Rev. Lett. **88**, 186402 (2002).
[11] See, e.g., J.-P. Blaizot and G. Ripka, *Quantum Theory of Finite Systems* (MIT Press, Cambridge, MA, 1986).
[12] T. P. Eggarter and R. Riedinger, Phys. Rev. B **18**, 569 (1978).
[13] A. Comtet et al., Ann. Phys. (N.Y.) **239**, 312 (1995).
[14] J. J. M. Verbaarschot and I. Zahed, Phys. Rev. Lett. **73**, 2288 (1994).
[15] K. Slevin and T. Nagao, Phys. Rev. Lett. **70**, 635 (1993).
[16] P. W. Brouwer et al., Phys. Rev. Lett. **81**, 862 (1998).
[17] H. Fukuyama and P. A. Lee, Phys. Rev. B **17**, 535 (1978).
[18] M. V. Feigelman, Sov. Phys. JETP **52**, 555 (1980).
[19] T. Giamarchi and H. J. Schultz, Phys. Rev. B **37**, 325 (1988).
[20] A. I. Larkin and Y. N. Ovchinnikov, Sov. Phys. JETP **38**, 854 (1974); Y. Imry and S.-K. Ma, Phys. Rev. Lett. **35**, 1399 (1975).
[21] M. Kardar et al., Phys. Rev. Lett. **56**, 889 (1986).
[22] A. M. Polyakov, Phys. Rev. E **52**, 6183 (1995); W. E et al., Phys. Rev. Lett. **78**, 1904 (1997).
[23] U. Buchenau et al., Phys. Rev. B **43**, 5039 (1991).