Calogero-Moser Models with Noncommutative Spin Interactions

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We construct integrable generalizations of the elliptic Calogero-Sutherland-Moser model of particles with spin, involving noncommutative spin interactions. The spin coupling potential is a modular function and, generically, breaks the global spin symmetry of the model down to a product of U(1) phase symmetries. Previously known models are recovered as special cases.

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1. Introduction.—The inverse-square interacting particle system [1–3] and its spin generalizations [4–9] are important models of many-body systems, due to their exact solvability and intimate connection to spin chain systems [10–14], two-dimensional Yang-Mills theories [15–17] etc. Reference [18] is a classic review, while [19,20,21] present newer different perspectives.

The prototype of these models is the spin-Calogero (''rational'') scattering model of particles on the line carrying U(n) spin and interacting with two-body inverse-square potentials with a U(n)-invariant spin coupling. Most other models can be obtained as appropriate reductions of this model, taking advantage of its discrete or continuous symmetries [22]. In particular, generalizations involving U(n) noninvariant interactions can be obtained this way, recovering the trigonometric models derived in [23,24] and extending them to the elliptic case [22].

An unrelated development has been the recent progress in noncommutative field theory and matrix models. Spatial noncommutativity can be traced back to Heisenberg and naturally arises in lowest Landau level physics [25]. Its current manifestation originates in matrix, string, and membrane theory [26,27] and came into focus with the work of Connes, Douglas, and Schwartz [28].

So far these two fields remained unrelated. In this Letter we show how they can be cross-fertilized by borrowing notions of noncommutative geometry and applying them in the reduction scheme of the Calogero model to obtain a new integrable elliptic model involving non-U(n) invariant noncommutative spin interactions. Such a modification of the spin interaction may serve to test the ''flavor stiffness'' of the original spin model, to stress the degeneracy structure of the energy spectrum and to identify universality features of this class of models.

2. The reduction scheme.—The basic technique that we will use consists of reducing a system of infinitely many particles with spin to a finite system with generalized interactions. The reader should refer to [22], and especially to the elliptic case with spin, for a more detailed description of the method.

The starting point is the spin-Calogero system with classical U(n) degrees of freedom. This system can be obtained, for instance, from the model in [4,5] (which can itself be obtained as a reduction of a Hermitian matrix model [29] into nontrivial angular momentum sectors) by redistributing the global U(N) degrees of freedom of this model into individual particle spins or, alternatively, from the infinite-volume classical limit of the spin model derived and solved in [16]. The Hamiltonian for N particles reads

$$H = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{\mathrm{tr}(S_i S_j)}{x_{ij}^2}. \tag{1}$$

x_i and p_i are one-dimensional canonical coordinates and momenta; S_i are a set of independent classical U(n) spins of rank one and length ℓ, that is, n $\times$ n rank-one Hermitian matrices satisfying

$$\mathrm{tr}(S_i)^2 = \ell^2 \tag{2}$$

and with Poisson brackets

$$\{(S_i)_{ab}, (S_j)_{cd}\} = -i\delta_{ij}[(S_i)_{ad}\,\delta_{cb} - \delta_{ad}\,(S_i)_{cb}]. \tag{3}$$

Such spins can be realized in terms of oscillators [16]:

$$(S_i)_{ab} = \bar{A}_i^a A_i^b, \qquad a, b = 0 \ldots \mathrm{n} - 1, \tag{4}$$

where $(A_i^a, \bar{A}_i^a)$ are a set of nN independent classical harmonic oscillator canonical pairs with Poisson brackets:

$$\{A_i^a, \bar{A}_j^b\} = i\delta_{ij}\,\delta_{ab}, \tag{5}$$

and satisfying the constraint

$$\sum_a \bar{A}_i^a A_i^a = \ell \quad \text{for all } i. \tag{6}$$

In the above model we can analytically continue the coordinates x_i and momenta $p_i = \dot{x}_i$ to the complex plane. The integrability and solvability of this model trivially extends to the complex case. Such extensions will be useful, provided that we can identify a real subsystem, which will be the physical system of interest.

The Hamiltonian H is invariant under particle permutations, coordinate space translations, and global spin rotations:

$$i \to p_N(i); \qquad x_i \to x_i + c; \qquad S_i \to \mathrm{U} S_i \mathrm{U}^{-1}, \quad (7)$$

where p_N is an element of the permutation group S_N, c is a constant complex parameter, and U is a constant U(n) matrix. We can arrange, therefore, for special initial configurations that are invariant under some combination of the above symmetries and be sure that these conditions will be preserved in time.

We shall choose the configuration to be a replication of N real coordinates over an infinite complex lattice with periods c_1 and c_2. The N particles effectively live on the real coordinate of a complex torus. The total number of particles on the covering space is, thus, infinite and we can parametrize them with the triplet of indices $(i; m, n)$, where $i = 1, \ldots N$ labels the particles in each cell and $m, n \in \mathbf{Z}$ parametrize the cell. Shifts in m and n are elements of the (infinite) permutation symmetry of the system. The kinematical variables are chosen to obey

$$x_{i;m+1,n} = x_{i;m,n} + c_1, \qquad p_{i;m+1,n} = p_{i;m,n}, \quad (8)$$

$$x_{i;m,n+1} = x_{i;m,n} + c_2, \qquad p_{i;m,n+1} = p_{i;m,n,} \quad (9)$$

ensuring that we are dealing with lattice copies. This means

$$x_{i;m,n} = x_i + mc_1 + nc_2, \qquad p_{i;m,n} = p_i. \quad (10)$$

The above conditions, being an invariance under combined permutations and translations, are dynamically preserved. To also preserve the condition that x_i are real, we should impose invariance under the imaginary parity transformation $x \to x^*$. Choosing c_1 real, the only possibilities for c_2 are

$$c_2 + c_2^* = 0 \quad \text{or} \quad \mathrm{c}_2 + c_2^* = c_1. \quad (11)$$

The first choice (c_2 imaginary) leads to an orthogonal lattice, while the second choice leads to a rhombic lattice.

To ensure full preservation of the lattice structure, we should also impose appropriate periodic conditions for the spins. In this, we can take advantage of the global spin symmetry of the system and impose

$$S_{i;m+1,n} = \mathrm{U} S_{i;m,n} \mathrm{U}^{-1} \qquad S_{i;m,n+1} = V S_{i;m,n} V^{-1}, \quad (12)$$

with U and V two constant matrices. That is, spins can pick up U(n) transformations as they move around the cycles c_1 or c_2. Consistency requires that $S_{i;m+1,n+1}$ be uniquely determined irrespective of the order of increase of the indices m and n. That is,

$$\mathrm{U} V S_{i;m,n} V^{-1} \mathrm{U}^{-1} = V \mathrm{U} S_{i;m,n} \mathrm{U}^{-1} V^{-1}, \quad (13)$$

which implies

$$[\mathrm{U}^{-1} V^{-1} \mathrm{U} V, S_{i;m,n}] = 0. \quad (14)$$

For this to hold for all generic S_i we must require $\mathrm{U}^{-1} V^{-1} \mathrm{U} V \equiv \omega$ to be proportional to the identity matrix. Clearly ω satisfies $\det(\omega) = \omega^{\mathrm{n}} = 1$, so we obtain

$$\mathrm{U} V = \omega V \mathrm{U}, \qquad \omega = e^{i2\pi(\nu/n)}, \quad (15)$$

with ν an integer $0 \le \nu < \mathrm{n}$. U and V then satisfy Weyl's braiding condition which characterizes a noncommutative ("quantum") torus [30].

The spin matrices $S_{i;m,n}$ are now expressed as

$$S_{i;m,n} = \mathrm{U}^m V^n S_i V^{-n} \mathrm{U}^{-m} = V^n \mathrm{U}^m S_i \mathrm{U}^{-m} V^{-n}. \quad (16)$$

Inserting the expressions (10) and (16) in the Hamiltonian (1) we obtain the Hamiltonian of the reduced system. As usual, the resulting Hamiltonian has an infinite factor, due to the summation of the Hamiltonians of the infinitely many identical cells over the complex plane. Dropping this trivial infinity, the reduced Hamiltonian includes the kinetic terms of the fundamental cell and the interaction potential of particles in this cell with all other particles in all cells:

$$H = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \frac{1}{2} \sum_{i,j} \sum_{m,n=-\infty}^{\infty} \frac{\mathrm{tr}(\mathrm{U}^m V^n S_i V^{-n} \mathrm{U}^{-m} S_j)}{(x_{ij} + mc_1 + nc_2)^2}, \quad (17)$$

where we adopted the notation $x_{ij} = x_i - x_j$.

3. Noncommutative spin interaction potentials.—To proceed, we must identify the possible forms of U, V. We need the irreducible representations of the relation (15). Call k the greatest common divisor of ν and n. Then $\mathrm{n} = k\mathrm{m}$ and $\nu = k\mu$, for relatively prime m, μ. The irreducible representations for U, V are m-dimensional "clock" and "shift" matrices. By a global U(n) spin transformation we can diagonalize either of U, V. Choosing U diagonal, the general form of U and V will be the direct sum of k of the above irreducible representations:

$$\begin{aligned} \mathrm{U} &= \mathrm{diag}\{e^{i\phi_0}, \ldots e^{i\phi_{k-1}}\} \otimes u, \\ V &= \mathrm{diag}\{e^{i\theta_0}, \ldots e^{i\theta_{k-1}}\} \otimes v, \end{aligned} \quad (18)$$

where ϕ_q, θ_q are arbitrary phases, determining the Casimirs U^{m} and V_{m}, and u, v are the m-dimensional clock and shift matrices

$$\begin{aligned} &u_{\alpha\beta} = \omega^\alpha \delta_{\alpha\beta}, \qquad v_{\alpha\beta} = \delta_{\alpha+1,\beta}(\mathrm{mod\,m}), \\ &\alpha, \beta = 0, \ldots \mathrm{m} - 1. \end{aligned} \quad (19)$$

So the acceptable U and V depend on $2k$ arbitrary parameters.

To take advantage of the form (18) for U, V we partition S_i into k^2 blocks of dimension $\mathrm{m} \times \mathrm{m}$ each by using the double index notation

$$(S_i)_{ab} = (S_i)^{pq}_{\alpha\beta}, \qquad a = p\mathrm{m} + \alpha, \qquad b = q\mathrm{m} + \beta. \quad (20)$$

The U(n) Poisson brackets in this notation are

$$\{(S_i)^{pq}_{\alpha\beta}, (S_j)^{rs}_{\gamma\delta}\} = -i\delta_{ij}[(S_i)^{ps}_{\alpha\delta} \delta_{\gamma\beta} \delta_{rq} - \delta_{\alpha\delta} \delta_{ps} (S_i)^{rq}_{\gamma\beta}]. \quad (21)$$

The m, n-sums that appear in (17) then become

$$\sum_{m,n;\alpha,\beta;p,q} (S_i)^{pq}_{\alpha+n,\beta+n} (S_j)^{qp}_{\beta,\alpha} \frac{e^{-im\phi_{pq}-in\theta_{pq}}\,\omega^{m(\alpha-\beta)}}{(x_{ij}+mc_1+nc_2)^2}, \tag{22}$$

where the term $m = n = 0$ is omitted if $i = j$.

The above gives a potential interaction between particles i and j in the form of a modular function in x_{ij} which depends on the spin components of particles i and j. To make the noncommutative character of the spin interaction manifest, we perform a change of basis in the spin states and define

$$(\tilde{S}_i)^{pq}_{\alpha\beta} = \sum_{\sigma} \omega^{[\sigma+(\alpha/2)]\beta} (S_i)^{pq}_{\alpha+\sigma,\sigma}. \tag{23}$$

This is essentially a discrete Fourier transform in the sum of the α, β indices of $S^{pq}_{\alpha\beta}$. (Note that, for m odd, $\tilde{S}^{pq}_{\alpha\beta}$ is actually antiperiodic in the index α if β is odd, and vice versa. Although we could have defined a properly periodic matrix, we prefer this slight inconvenience in order to make the ensuing formulas more symmetric.) In fact, it will be convenient to assemble the double indices (α, β) and (m, n) into vectors $\vec{\alpha}$ and $\vec{m}$. Similarly, we define $\vec{c} = (c_1, c_2)$ and $\vec{\phi}_p = (\phi_p, \theta_p)$.

The Poisson brackets of the $\tilde{S}_i$ are found from (21)

$$\{(\tilde{S}_i)^{pq}_{\vec{\alpha}}, (\tilde{S}_j)^{rs}_{\vec{\beta}}\} = i\delta_{ij}[\omega^{(\vec{\alpha}\times\vec{\beta})/2}\,\delta_{ps}\,(\tilde{S}_i)^{rq}_{\vec{\alpha}+\vec{\beta}} - \omega^{-(\vec{\alpha}\times\vec{\beta})/2}\,\delta_{rq}\,(\tilde{S}_i)^{ps}_{\vec{\alpha}+\vec{\beta}}]. \tag{24}$$

This is a structure extending the Moyal (star-commutator) algebra, the exponent of ω being the cross product of the discrete "momenta" $\vec{\alpha}$ and $\vec{\beta}$. For $(rs) = (pq)$, in particular, it becomes the torus Fourier transform of the Moyal bracket

$$\{(\tilde{S}_i)^{pq}_{\vec{\alpha}}, (\tilde{S}_j)^{pq}_{\vec{\beta}}\} = i\delta_{ij}[\omega^{1/2} - \omega^{-(1/2)}][\vec{\alpha}\times\vec{\beta}]_\omega (\tilde{S}_i)^{pq}_{\vec{\alpha}+\vec{\beta}}, \tag{25}$$

where

$$[x]_\omega = \frac{\omega^{x/2} - \omega^{-(x/2)}}{\omega^{1/2} - \omega^{-(1/2)}} \tag{26}$$

is the ω-deformation of x. This is the so-called trigonometric algebra with periodic discrete indices [31].

Finally, by inverting (23) and substituting in (22), the potential energy W in terms of the $\tilde{S}_i$ acquires the form

$$W = \sum_{i,j} \sum_{\vec{\alpha};p,q} (\tilde{S}_i)^{pq}_{\vec{\alpha}} (\tilde{S}_j)^{qp}_{-\vec{\alpha}} W^{pq}_{\vec{\alpha}}(x_{ij}). \tag{27}$$

The above includes two-body interactions, for $i \neq j$, as well as spin self-couplings, for $i = j$, arising from the interaction of each particle with its own images in different cells. The two-body potential $W^{pq}_{\vec{\alpha}}(x)$ is

$$W^{pq}_{\vec{\alpha}}(x) = \frac{1}{m}\sum_{\vec{m}} \frac{\omega^{\vec{\alpha}\times\vec{m}}\, e^{i\vec{\phi}_{pq}\cdot\vec{m}}}{(x+\vec{c}\cdot\vec{m})^2}, \tag{28}$$

while the spin self-coupling $\tilde{W}^{pq}_{\vec{\alpha}}$ is

$$\tilde{W}^{pq}_{\vec{\alpha}} = \frac{1}{m}\sum_{\vec{m}\neq\vec{0}} \frac{\omega^{\vec{\alpha}\times\vec{m}}\, e^{i\vec{\phi}_{pq}\cdot\vec{m}}}{(\vec{c}\cdot\vec{m})^2} = \lim_{x\to 0}\Big[W^{pq}_{\vec{\alpha}}(x) - \frac{1}{mx^2}\Big]. \tag{29}$$

If the above potentials were independent of the U(n) indices $\vec{\alpha}$ and p, q, the sum over U(n) indices in the potential energy expression (27) would simply be a U(n) trace and would give the U(n)-invariant coupling between the spins of particles i and j multiplying the standard Weierstrass potential of the elliptic Calogero model. In the present case, however, the above potential is spin-dependent and breaks U(n) invariance, introducing a star-product twist in the indices $\vec{\alpha}$ and phase shifts $\vec{\phi}_p$ in the indices p, q. Generically, the U(n) invariance of the original model is broken down to an Abelian $U(1)^k$, amounting to the transformation

$$(S_i)^{pq}_{\alpha\beta} \to e^{i\varphi_p} (S_i)^{pq}_{\alpha\beta} e^{-i\varphi_q}. \tag{30}$$

If $\vec{\phi}_p$ are equal for k' values of p, the remaining symmetry $U(1)^{k'}$ is enhanced to $U(k')$, corresponding to mixing the corresponding p-components.

The case $\omega = 1$, $\vec{\phi}_p = 0$ reduces to the standard spin-elliptic Calogero-Moser model. The case $m = 1$ (and thus $\omega = 1$) reproduces the U(n)-noninvariant model introduced in [22]. The general case with $\omega \neq 1$ is a new classical integrable model of the spin-Calogero type.

The potentials can be expressed in terms of theta-functions. $W^{pq}_{\vec{\alpha}}(x)$ is a modular function on the complex torus (c_1, c_2) with quasiperiodicity

$$\begin{aligned} W^{pq}_{\vec{\alpha}}(x+c_1) &= e^{-i\phi_{pq}+[(2\pi\mu)/m]\alpha_2} W^{pq}_{\vec{\alpha}}(x), \\ W^{pq}_{\vec{\alpha}}(x+c_2) &= e^{-i\theta_{pq}-[(2\pi\mu)/m]\alpha_1} W^{pq}_{\vec{\alpha}}(x). \end{aligned} \tag{31}$$

It has a double pole at $x = 0$, with principal part

$$W^{pq}_{\vec{\alpha}}(x) = \frac{1}{mx^2} + O(x^0) \tag{32}$$

and no other poles in each cell. These properties uniquely define $W^{pq}_{\vec{\alpha}}(x)$ and allow for an expression in terms of theta-functions. We put

$$W^{pq}_{\vec{\alpha}}(x) = A\omega^{-i(x/c_1)} e^{-i(x/c_1)\phi_{pq}} \times \frac{\vartheta_1[\frac{\pi}{c_1}(x-Q_1)]\vartheta_1[\frac{\pi}{c_1}(x-Q_2)]}{\vartheta_1(\frac{\pi}{c_1}x)^2}, \tag{33}$$

where $Q_{1,2}$ are the as yet unknown zeros of $W^{pq}_{\vec{\alpha}}(x)$ and the theta-functions appearing above have complex period $T = c_2/c_1$. This has the right quasiperiodicity under $x \to x + c_1$. In order to also have the right quasiperiodicity under $x \to x + c_2$, $Q_{1,2}$ must satisfy

$$Q_1 + Q_2 = \frac{\vec{\phi}_{ab}\times\vec{c}}{2\pi} + \frac{\mu}{m}\vec{\alpha}\cdot\vec{c} \tag{34}$$

and to have the right behavior around $x = 0$ we must further have

$$\frac{\vartheta_1'(\frac{\pi}{c_1}Q_1)}{\vartheta_1(\frac{\pi}{c_1}Q_1)}+\frac{\vartheta_1'(\frac{\pi}{c_1}Q_2)}{\vartheta_1(\frac{\pi}{c_1}Q_2)}=-i\frac{\phi_{ab}}{\pi}-2i\frac{\mu}{\mathrm{m}}\alpha_1, \qquad (35)$$

$$A=\frac{\pi^2\,\vartheta_1'(0)^2}{\mathrm{m}c_1^2\,\vartheta_1(\frac{\pi Q_1}{c_1})\vartheta_1(\frac{\pi Q_2}{c_1})}. \qquad (36)$$

Equations (34) and (35) above determine Q_1 and Q_2, while (36) in turn determines A.

The sums appearing in (28) and (29) are in general convergent, due to the presence of the phases. For $\omega = 1$, however, the phases are absent and terms with $p = q$ have an additive ambiguity due to the need for regularization for the expression (28). In the theta-function expression this manifests in the fact that the equations for $Q_{1,2}$ (34) and (35) are satisfied for *any* $Q_1 = -Q_2$. By applying the addition formula

$$\vartheta_1(x+Q)\vartheta_1(x-Q)\vartheta_4(0)^2=\vartheta_1(x)^2\vartheta_4(Q)^2-\vartheta_4(x)^2\vartheta_1(Q)^2 \qquad (37)$$

this is seen indeed to amount to an arbitrary additive constant to the expression for $W^{pp}(x)$. The same holds for terms p, q for which $\tilde{\phi}_{pq}=0$. Such arbitrariness, however, corresponds to trivial redefinitions of the model by addition of constants of motion. This is explained in [22] and will not be elaborated here.

4. Conclusions and open questions.—In conclusion, we identified an integrable generalization of the elliptic spin model which breaks the spin U(n) invariance and promotes the potential to a modular function introducing noncommutative spin twists. (Quantum generalizations of elliptic spin models have appeared recently, but with U(n) invariant interactions [32].)

There are clearly many issues that deserve further study. The conserved quantities and Lax matrix of this model can, in principle, be obtained from the corresponding quantities of the unreduced model; this was done, e.g., in [22] for a specific case. A derivation in the present case would be useful. Further, the modular potential involves implicitly defined $Q_{1,2}$; a more explicit and symmetric expression would be desirable.

The properties of the spin interaction should also be clarified. In particular, it would be interesting to see if some deformation of U(n) can be identified as a dynamical symmetry.

Finally, the quantum mechanical extension of the model is, perhaps, the most interesting and pressing question. This is not trivially accessible by the method used here since, in general, the constraints implied by the phase space restrictions are second class and we cannot carry over the solution of the unrestricted quantum model and apply the constraints as operator relations on the Hilbert space. The above issues remain interesting topics for future investigation.

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