Dynamic Critical Behavior Near the Superfluid Transition in ^{3}He-^{4}He Mixtures in Two Loop Order

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We calculated in two loop order the field theoretic renormalization group functions taking into account the decomposition of the dynamical vertex functions into the static vertex functions and genuine dynamical parts. The observation of this nonperturbative structure simplifies the theoretical expressions obtained by perturbation theory considerably and makes tractable a complete two loop calculation of the critical dynamics near the superfluid transition of ^{3}He-^{4}He mixtures (model F'). As a result, we obtain various transport coefficients, which govern the nonasymptotic and nonuniversal temperature dependence. We also correct long-standing results for the critical dynamics of the superfluid transition in pure ^{4}He (model F) and for the dynamics of structural or magnetic phase transitions (model C).

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Universality in dynamical critical phenomena has led to a series of models [1] labeled by the capital letters between A and J, which describe the critical dynamics in different physical systems such as ferromagnets and antiferromagnets, pure and binary fluids, and superfluids. Comparison with experiment has reached a high level for the gas-liquid transition and most prominently for the superfluid transition. For both systems, zero gravity experiments have been performed (see, e.g., NASA project "Fundamental Physics in Space" [2]). It has turned out that for several different reasons two loop calculations are necessary on the theoretical side to reach the experimental accuracy (both for universal properties as well as for the nonasymptotic nonuniversal behavior). This holds for the measurements of transport coefficients such as the shear viscosity in fluids [3] and the thermal conductivity in ^{4}He near the superfluid transition [4]. In the latter case, a complete two loop field theoretic renormalization group calculation has been performed [5].

The situation in ^{3}He-^{4}He mixtures was less favorable although a lot of experimental results mostly by the group of Meyer are available (see references in [6]). Thus far, the critical dynamics has been treated [6] in a combination of a one loop model with static couplings to the secondary densities of entropy and concentration (model F') and the two loop terms without these couplings (model E' [7]). This approximation was used for a comparison with the temperature dependence of the three transport coefficients—thermal conductivity, thermal diffusion ratio, and mass diffusion. The interrelation between the three coefficients allows a much better significant test of the theory.

In order to improve the situation, one has to look for a new input into the straightforward but from the amount of terms exploding loop expansion. We have reached this by (i) separating strictly statics from dynamics within the expression for the vertex functions and (ii) introducing "summed up" static quantities such as the correlation length as performed in [8]. This makes the calculation tractable, and especially item (i) leads to further checks of the calculation and thus a high probability of the correctness of the results obtained.

Thus, we achieve complete consistent two loop order results for model F' and as a corollary for the field theoretic functions for model F [5] and model C (anisotropic magnets and structural transitions) [9]. These results are corrected by our calculations, while for model C they agree with those in Ref. [10]. The general method presented in this Letter has also been applied to calculate the field theoretic functions for other models [11,12].

Let us consider densities $a_i(x)$ which may be the order parameter ϕ and other conserved densities α_j necessary for describing the dynamics of the slow system considered. Then the dynamic equations of the Ma and Mazenko type [13], which are usually used in dynamic renormalization group theory, have the form

$$\frac{\partial a_i}{\partial t} = \sum_j [k_B T\{a_i, a_j^*\} - L_{ij}] \frac{\delta H(\{a\})}{\delta a_j^*} + \theta_i. \tag{1}$$

The first sum at the right-hand side of the above equation contains the reversible contributions, which are determined by a set of generalized Poisson brackets $\{a_i, a_j^*\}$ between the densities. The second term represents the dissipative part determined by a set of Onsager coefficients L_{ij}, where $L_{ij} = \Lambda_{ij}$ for nonconserved densities and $L_{ij} = -\Lambda_{ij}\nabla^2$ for conserved densities. The θ_i are stochastic forces. $H(\{a\})$ is a convenient static functional including the static critical behavior of the considered system. For instance, in models A, B, and J (the order parameter is the only density, see for notation [1]), this is the well-known ϕ^4 Ginzburg-Landau-Wilson functional with the fourth order coupling u. The other models include additional conserved densities of Gaussian order (models E, G, and H) or additional cubic terms γ between the square of the order parameter and the other densities (models C, F, and F'). With methods of group theory, it is possible to introduce generalized Poisson brackets in a systematic manner [14]. The Poisson brackets are defined in a way that the corresponding reversible hydrodynamic equations are reproduced. In the same way, the Onsager coefficients are introduced in accordance to the considered hydrodynamics. With a Gaussian static functional $H = \int d^dx \frac{1}{2}\sum_{ml} a_m(x)\chi_{ml}a_l(x)$, Eq. (1) reproduces the hydrodynamic equations. Additional nonlinear terms enter Eq. (1) only via higher order terms in the static functional. Thus, the hydrodynamic structure in the brackets in (1) is also conserved with these contributions. Hence, we expect that the hydrodynamic basic structure is conserved in the Fourier transformed dynamic vertex functions when the statics is treated appropriately. More specifically (for notation and definitions, see [15]), we expect that the separate appearance of the static vertex function in the dynamic vertex function known from zero and one loop order is valid also in higher orders, and one can write them most generally as functions of the correlation length ξ, the wave vector k, and the frequency ω:

$$\Gamma_{a_i\tilde{a}_j}(\xi,k,\omega) = -i\omega\Omega_{a_i\tilde{a}_j}(\xi,k,\omega) + \sum_m \Gamma^{(st)}_{a_ia_m}(\xi,k)\Gamma^{(d)}_{a_m\tilde{a}_j}(\xi,k,\omega), \quad (2)$$

with the static vertex function $\Gamma^{(st)}_{a_ia_m}$. This structure justifies the method used thus far to identify the hydrodynamic transport coefficients with the vertex functions. Indeed, we verified the above structure of the vertex functions in two loop calculations for all models (A to J) mentioned. We have performed the dynamic perturbation expansion and used then exact algebraic rearrangements of the expressions to obtain the structure in (2). We want to emphasize that these rearrangements are quite nontrivial and that they require some calculational expense since the structure is not valid for the topological different contributions of the loop expansion. Only the sum of all contributions has this structure. We then obtain the two functions $\Omega_{a_i\tilde{a}_i}$ and $\Gamma^{(d)}_{a_m\tilde{a}_j}$ in a relatively lucid form.

In addition, general relations for other vertex functions, necessary for the calculation of the dynamic scattering functions, could be proofed in two loop order,

$$\Gamma_{\tilde{\phi}\tilde{\phi}}(\xi,k,\omega) = -2\mathrm{Re}[\Omega_{\phi\tilde{\phi}}(\xi,k,\omega)\Gamma^{(d)}_{\phi\tilde{\phi}}(\xi,k,\omega)] \quad (3)$$

and for the diagonal part of the secondary densities,

$$\Gamma_{\tilde{\alpha}_i\tilde{\alpha}_i}(\xi,k,\omega) = -2\mathrm{Re}\left[\sum_l \Omega_{\alpha_l\tilde{\alpha}_i}(\xi,k,\omega)\Gamma^{(d)}_{\alpha_l\tilde{\alpha}_i}(\xi,k,\omega)\right]. \quad (4)$$

No such relation could be found for the nondiagonal parts, which turn out to be complex functions.

These genuine dynamic functions contain singularities not only at the critical dimension d_c, but they also contain poles at lower dimensions (e.g., $d = 3$) due to a shift of T_c and the expansion of the correlation length ξ. These poles are purely static and can be eliminated by resumming the T_c shift and the correlation length. This procedure extends the method introduced by Schloms and Dohm for static vertex functions [8] to dynamics and guarantees regular expressions at $d = 3$ independent of the renormalization method used.

The big advantage of using the above structure (2) and the resummation is that the former huge amount of integral expressions, which has to be calculated, is reduced to a minimal number. This makes it possible to calculate even models which have been considered as too extensive in past years (such as model F' for ^{3}He-^{4}He mixtures) in two loop order. Even in these most complex models, only eight independent integrals remain for an explicit calculation after the rearrangement.

The different dynamic functions in (2) have a specific dependence on the set of couplings leading to obvious simplifications for models without the static couplings $\{\gamma\}$ and/or dynamic couplings $\{g\}$,

$$\Omega_{\phi\tilde{\phi}} = 1 + \Omega_A(u,\Gamma) + \sum_i \gamma_i W^i_{\phi\tilde{\phi}}(u,\{\gamma\},\{g\},\Gamma,\{\Lambda\}), \quad (5)$$

$$\Omega_{\alpha_i\tilde{\alpha}_j} = \delta_{ij} + \gamma_i W_{\alpha_i\tilde{\alpha}_j}(u,\{\gamma\},\{g\},\Gamma,\{\Lambda\}), \quad (6)$$

$$\Gamma^{(d)}_{\phi\tilde{\phi}} = \Gamma k^a + \sum_i g_i G^i_{\phi\tilde{\phi}}(u,\{\gamma\},\{g\},\Gamma,\{\Lambda\}), \quad (7)$$

$$\Gamma^{(d)}_{\alpha_i\tilde{\alpha}_j} = \Lambda_{ij}k^2 + g_i G^i_{\alpha_i\tilde{\alpha}_j}(u,\{\gamma\},\{g\},\Gamma,\{\Lambda\}) \quad (8)$$

(we suppressed the dependence on ξ, k, and ω of all these functions), where $a = 2$ for a conserved and $a = 0$ for a nonconserved order parameter. The term $\Omega_A(u,\Gamma)$ always appears in (3). Its singular contribution depends on the conservation property of the order parameter, which is nonconserved for model A and conserved for model B (no new dynamic singularities). Note that all the relations mentioned thus far are valid in the unrenormalized perturbation theory.

All of our following calculations are performed within the field theoretical formulation of renormalization group theory [15]. The calculated dynamic functions Ω and $\Gamma^{(d)}$ contain now singularities at d_c ($d_c = 4$ apart from model J, where $d_c = 6$) which will be absorbed into dynamic Z factors and which determine the dynamic exponents. For a conserved order parameter, only $\Gamma^{(d)}$ contain poles which need an independent dynamic renormalization. This is also true for all secondary densities because they are in any case conserved quantities. For a nonconserved order parameter both functions need dynamic Z factors.

We now present the two loop result for the ζ functions in model F' (all parameters are renormalized ones, see [6]).

$$\zeta_\Gamma = \sum_j \mathcal{F}_j^2 - \frac{2}{3}\sum_j u\mathcal{F}_j a_j - \frac{1}{2}\sum_{j,m} \mathcal{F}_j\mathcal{F}_m b_{jm} + \frac{u^2}{9}\left(L_0 + x_1 L_1 - \frac{1}{2}\right), \tag{9}$$

$$\zeta_{\lambda_j} = \gamma_j^2 - \frac{F_j^2}{2w_j'}\left(1 + \frac{1}{2}\mathrm{Re}[Q]\right), \tag{10}$$

where we have introduced the complex coupling $\mathcal{F}_j = C_j - i\mathcal{E}_j$ which is composed from couplings appearing in models C and E defined by

$$C_j = \sqrt{\frac{w_j}{1 + w_j}}\gamma_j, \qquad \mathcal{E}_j = \frac{F_j}{\sqrt{w_j(1 + w_j)}}. \tag{11}$$

In (8) and (9), we have defined the complex time scale ratios $w_j = \Gamma/\lambda_j = w_j' + iw_j''$ between the order parameter and the secondary density Onsager coefficients λ_j and the mode couplings $F_j = g_j/\lambda_j$ quite analogous to model F in pure ^{4}He [5]. The sum is running over the number of all secondary densities. The quantities a_j and b_{jk} in (7) are defined as

$$a_j = C_j(1 - x_1 L_1) + i\mathcal{E}_j x_- x_1 L_1 - \mathcal{F}_j L_0, \tag{12}$$

$$b_{jm} = C_j C_m(1 - 2x_1 L_1) + (C_j i\mathcal{E}_m + C_m i\mathcal{E}_j)(1 + x_- x_1 L_1) + \mathcal{E}_j\mathcal{E}_m\left\{[x_+ + v + x_+^2(x_+^2 + 2v^2)]\frac{L_1}{x_+} - 3v\right\} - 2\mathcal{F}_j\mathcal{F}_m L_0 - \frac{\mathcal{F}_j\mathcal{F}_m}{1 + w_j}[w_j + w_m^2 l_{mj}^{(a)} - w_j^2 l_{jm}^{(a)} + (1 + w_j - w_m)(1 + w_j + w_m) l_{jm}^{(s)}], \tag{13}$$

where we have introduced the ratio $v = \Gamma/\Gamma^*$. Further, the following definitions have been used: $x_\pm = 1 \pm v$, $x_1 = 2 + v$, and

$$L_0 = 2\ln\frac{2}{1 + v^{-1}}, \qquad L_1 = \ln\frac{(1 + v^{-1})^2}{1 + 2v^{-1}}, \tag{14}$$

$$l_{jm}^{(a)} = \ln\frac{1 + w_j}{1 + \frac{w_j}{w_m}}, \qquad l_{jm}^{(s)} = \ln\frac{(1 + w_j)(1 + w_m)}{1 + w_j + w_m}. \tag{15}$$

The function Q introduced in (8) is given by

$$Q = \sum_l \frac{w_l}{w_l'}\mathcal{F}_l\left[\mathcal{F}_l\left(\frac{1}{2} + \ln\frac{1 + w_l}{1 + w_l^*}\right) + \sqrt{\frac{w_l^*}{w_l}}\mathcal{F}_l^*|1 + w_l| - \left(\frac{W_l}{w_l}C_l + w_l i\mathcal{E}_l\right)W_l L_l\right], \tag{16}$$

with the quantities $W_l = w_l + w_l^* + w_l w_l^*$ and $L_l = \ln(1 + W_l^{-1})$.

Setting the static couplings γ_i equal to zero, one recovers model E' [7]. More important is the reduction to model F, which is found by just skipping the indices and the sums. In order to obtain model C, one sets equal to zero in model F the mode coupling F and the imaginary part w''. Regarding model F, the corrections we found with respect to Ref. [5] concern terms, which go to zero at the fixed point; thus, our results agree with model E [16].

In comparing our results with measurements of the hydrodynamic transport coefficients—thermal conductivity, thermal diffusion ratio, and mass diffusion—we proceed along the lines of Ref. [6]. The transport coefficients are determined by k^2 terms of $\Gamma^{(d)}_{\alpha_i\tilde{\alpha}_j}(\xi, k, \omega = 0)$ as functions of the static and dynamic parameters γ_i, u, F_i, and w_i.

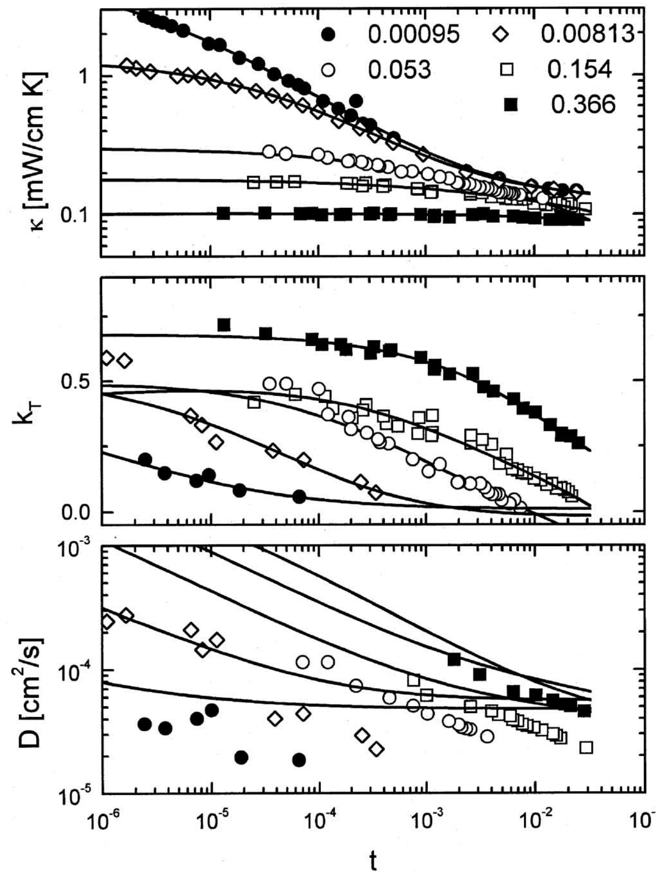

FIG. 1. Temperature dependence [$t = (T - T_c)/T_c$] of the thermal conductivity κ, the thermal diffusion ratio k_T, and the mass diffusion D in the mixture for different concentrations (solid curves theory, experimental data from [17,18] molar concentration as indicated in the figure). From a fit of the first two coefficients, the third is predicted.

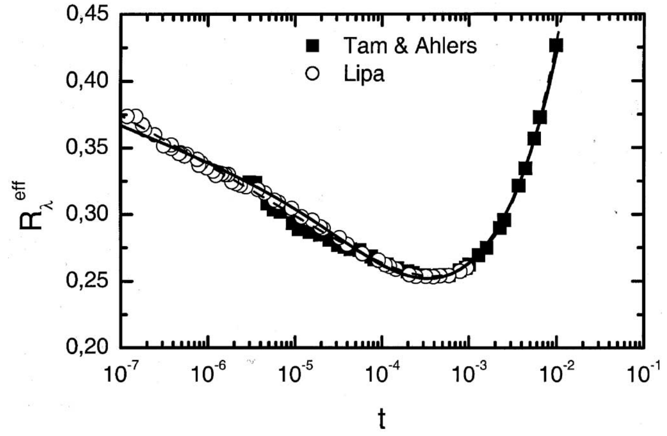

FIG. 2. Fit (full curve) of the amplitude R_λ at SVP of the thermal conductivity in the range 10^{-3} to 10^{-5} to the data of [19] and above 10^{-3} to the data of [20]. Dashed curve from [5].

They also have been calculated in two loop order making use of the results for the integrals already found in model *F* [5]. The modification of the expressions for the transport coefficients by the two loop terms has already been indicated by a function *P* [6] and turn out to be numerically small contributions. The temperature dependence of the dynamical parameters is known from flow equations determined by the ζ functions, Eqs. (7) and (8). Taking all these results into account, we fit the temperature dependence of two transport coefficients (thermal conductivity and thermal diffusion ratio) with the background values of the dynamic couplings and time ratios as adjustable parameters and predict the third one (mass diffusion) (see Fig. 1). There is (i) an improvement in the quality of the fit, (ii) some improvement in the prediction of the mass diffusion, but an obvious discrepancy remains, and (iii) the background values for the imaginary parts of the time ratios change to more reliable values.

We also have compared with the thermal conductivity $\lambda_T(t)$ at saturated-vapor-pressure conditions (SVP) in pure ^{4}He, respective with the effective amplitude $R_\lambda = \lambda_T/\sqrt{\xi g_0^2 k_B C_P}$ (see Fig. 2), with C_P the temperature dependent specific heat, g_0 the unrenormalized mode coupling and k_B the Boltzmann constant. The background parameter for the renormalized imaginary part of the time ratio w found from a fit with the correct flow equations is now of the expected size $w'' \approx 0.3$ instead of $w''_{\mathrm{Dohm}} \approx 0.8$. In Ref. [21], the unrenormalized value was shown to be approximately $w''_0 \approx 0.21$.

The next step concerning the suprafluid transition in the mixture would be (i) to improve the accuracy of various static and dynamic quantities entering the comparison with theory, and (ii) a comparison with sound velocity and sound absorption measurements both for pure ^{4}He and for the mixtures. For all other models, two loop calculations of dynamic amplitude ratios come now within the realm of possibility.

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