Universality of Small Scale Turbulence

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(Received 20 September 2001; published 3 September 2002)

The proposed universality of small scale turbulence is investigated for a set of measurements in a cryogenic free jet with a variation of the Reynolds number (Re) from 8500 to 10^6 ($\max R_\lambda \approx 1200$). The traditional analysis of the statistics of velocity increments by means of structure functions or probability density functions is replaced by a new method which is based on the theory of Markov processes. It gives access to a more complete characterization by means of joint probabilities of finding velocity increments at several scales. Based on this more comprehensive method, our results are very far from a possible universal state, even for R_λ above 1000.

DOI: 10.1103/PhysRevLett.89.124502 PACS numbers: 47.27.Ak, 05.10.Gg, 47.27.Jv

The complex behavior of turbulent fluid motion has been the subject of numerous investigations over the past 60 years and still the problem is not solved [1]. Especially the unexpected frequent occurrences of high values for velocity fluctuations on small scales, known as small scale intermittency, remain a challenging subject for further investigations. The characterization of this remarkable feature of turbulence has led to many fruitful new concepts, which enabled us to get a deeper physical understanding of many complex systems.

Based on the pioneering works [2–4], turbulence is usually assumed to be universal, i.e., for scales r within the inertial range $\eta \ll r \ll L$, the statistics of the velocity field is independent of the large scale boundary conditions, the mechanism of energy dissipation, and the Reynolds number (Re). L denotes the integral length and η the dissipation length. The assumed universality has gained considerable importance for models and numerical methods such as large eddy simulations; cf. [5]. Finding experimental evidence for the validity of the assumed universality is therefore of utmost importance.

A central quantity to be investigated is the so-called longitudinal velocity increment $u(r)$,

$$u(r) = \mathbf{e} \cdot [\mathbf{v}(\mathbf{x} + \mathbf{e}r, t) - \mathbf{v}(\mathbf{x}, t)], \tag{1}$$

where r is a certain length scale, $\mathbf{v}$ and $\mathbf{e}$ denote the velocity and a unit vector with arbitrary direction, respectively. Traditionally, the statistics of $u(r)$ is characterized by its moments $S_u^n(r) = \langle u^n(r) \rangle$, the so-called structure functions. Within the inertial range, proposed self-similarity leads to $S_u^n(r) \propto r^{\zeta_n}$. More pronounced scaling behavior is found for the so-called extended self-similarity method [6].

Experimental investigations carried out in several flow configurations at a large variety of Re numbers yield strong evidence that the scaling exponents ζ_n, in fact, show universal behavior, independent of the experimental setup [7]. A different result, however, was found for the probability density functions (pdf) $p(u, r)$. Studies using the theoretical framework of infinitely divisible multiplicative cascades show that the relevant parameters describing intermittency depend on the Re number [8].

From the point of view of statistics, a characterization of the scale dependent disorder of turbulence by means of structure functions or pdfs $p(u, r)$ is incomplete. Theoretical studies [9] point out that a complete statistical characterization of the turbulent cascade has to take into account the joint statistical properties of several increments on different length scales. An experimental study concerned with the statistical properties of small scale turbulence and its possible universalities therefore requires an analyzing tool which is not based on any assumption on the underlying physical process and which is capable of describing the multiscale statistics of velocity increments. Such a tool is given by the mathematics of Markov processes. This tool allows one to derive the stochastic differential equations governing the evolution of the velocity increment u in the scale parameter r from experimental data [10,11].

In this Letter we present, first, our new method to analyze experimental data, second, results for different Re numbers, and third, experimental findings which question the proposed universality.

The stochastic process governing the scale dependence of the velocity increment is Markovian, if

$$p(u_1, r_1 \mid u_2, r_2; \ldots ; u_N, r_N) = p(u_1, r_1 \mid u_2, r_2), \tag{2}$$

holds [12,13]. The conditional pdf $p(u_1, r_1 \mid u_2, r_2; \ldots ; u_N, r_N)$ describes the probability for finding the increment u_1 on the smallest scale r_1 provided that the increments $u_2, \ldots, u_N$ are given at the larger scales $r_2, \ldots, r_N$. We use the conventions $r_i \le r_{i+1}$ and $u_i = u(r_i)$. In [11,14] it has been shown that experimental data satisfy Eq. (2) if r_i and differences of scales $\Delta r = r_{i+1} - r_i$ are larger than an elementary step size l_{Mar},

 0031-9007/02/89(12)/124502(4)$20.00

comparable to the mean free path of molecules undergoing a Brownian motion.

As a consequence of (2), the joint pdf of N increments on N different scales simplifies to $p(u_1, r_1; u_2, r_2; \ldots; u_N, r_N) = p(u_1, r_1 \mid u_2, r_2) \ldots p(u_{N-1}, r_{N-1} \mid u_N, r_N) p(u_N, r_N)$. This indicates the importance of the Markov property: The entire information, i.e., any N-scale distribution of the velocity increment, is encoded in the conditional pdf $p(u, r \mid u_0, r_0)$ (with $r \leq r_0$).

For Markov processes the evolution of $p(u, r \mid u_0, r_0)$ in r is described by the Kramers–Moyal-expansion [12]. For turbulent data it was verified [11] that this expansion can be reduced to the Fokker-Planck equation:

$$-r\frac{\partial}{\partial r} p(u, r \mid u_0, r_0) = -\frac{\partial}{\partial u}[D^{(1)}(u, r) p(u, r \mid u_0, r_0)] + \frac{\partial}{\partial u^2}[D^{(2)}(u, r) p(u, r \mid u_0, r_0)]. \quad (3)$$

It is easily seen that the single scale pdf $p(u, r)$ obeys the same equation. Furthermore, the coefficients $D^{(1)}$ and $D^{(2)}$ (drift and diffusion coefficient, respectively) can be extracted from experimental data in a parameter-free way by their mathematical definition, see [12,13]:

$$D^{(k)}(u, r) = \lim_{\Delta r \to 0} \frac{r}{k! \Delta r} M^{(k)}(u, r, \Delta r), \quad (4)$$

$$M^{(k)}(u, r, \Delta r) = \int_{-\infty}^{+\infty} (\tilde{u} - u)^k p(\tilde{u}, r - \Delta r \mid u, r) d\tilde{u}. \quad (5)$$

Next, we focus on the analysis of experimental data measured in a cryogenic axisymmetric helium gas jet at Re numbers ranging from 8500 to 757 000. Each data set contains 1.6×10^7 samples of the velocity measured in the center of the jet in a vertical distance of $40D$ from the nozzle using a self-made hotwire anemometer ($D = 2$ mm is the diameter of the nozzle). Taylor's hypothesis of frozen turbulence was used to convert time lags into spatial displacements. Following the convention chosen in [11], the velocity increments are given in units of $\sigma_L = \sqrt{2}\sigma$, where σ is the standard deviation of the velocity fluctuations of the respective data set.

In order to check consistency of the data with commonly accepted features of fully developed turbulence, we calculated the dependence of the Taylor-scale Re number R_λ on the nozzle-based Re number. Figure 1 shows that R_λ scales like the square root of Re, in accordance with theoretical considerations and earlier experimental results; for further details, see [15].

The Markov condition (2) was checked using the method proposed in Ref. [11]. For all the data sets, the Markov property was found to be valid for scales r_i and differences of scales $\Delta r = r_{i+1} - r_i$ larger than the step size l_{Mar}, which turned out to be about equal to the Taylor microscale λ (for all Re numbers).

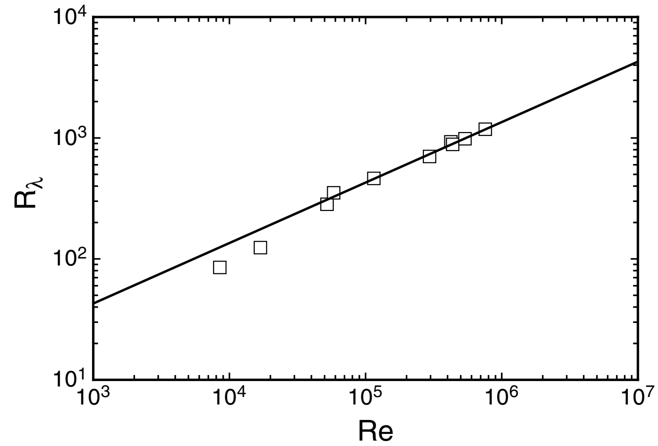

FIG. 1. Taylor-scale Reynolds number R_λ (for details on the determination, see [11]) as a function of the nozzle-based Reynolds number Re. Line: $R_\lambda = 1.35\sqrt{\mathrm{Re}}$.

Having determined the Markov length l_{Mar}, the coefficients $D^{(1)}(u, r)$ and $D^{(2)}(u, r)$ can be estimated from the measured conditional moments $M^{(1)}$ and $M^{(2)}$ according to Eq. (4). The extrapolation towards $\Delta r = 0$ was performed fitting linear functions to the measured $M^{(k)}$ in the interval $l_{\mathrm{Mar}} \leq \Delta r \leq 2 l_{\mathrm{Mar}}$ [16]. Figure 2 shows the resulting $D^{(1)}$ and $D^{(2)}$ for the data set at $R_\lambda = 1180$. The coefficients exhibit linear and quadratic dependencies on the velocity increment, respectively:

$$D^{(1)}(u, r) = -\gamma(r) u,$$
$$D^{(2)}(u, r) = \alpha(r) - \delta(r) u + \beta(r) u^2. \quad (6)$$

Equation (6) is found to describe the dependence of the $D^{(k)}$ on u for all scales r as well as for all Reynolds numbers investigated. By fitting the coefficients $D^{(k)}$ according to (6), it is thus possible to determine the scale dependence of the coefficients γ, α, δ, and β.

For α and δ, we obtain (see the inset of Fig. 3):

$$\alpha(r) = \alpha_0 \frac{r}{\lambda}, \qquad \delta(r) = \delta_0 \frac{r}{\lambda}. \quad (7)$$

The slopes α_0 and δ_0 strongly depend on the Re number (see Fig. 3) and can be approximated by:

$$\alpha_0 \approx 2.8 \mathrm{Re}^{-3/8}, \qquad \delta_0 \approx 0.68 \mathrm{Re}^{-3/8}. \quad (8)$$

For $\gamma(r)$ of $D^{(1)}$ (see Fig. 4), a universal function of r/λ is found which is well described by

$$\gamma(r) = \frac{2}{3} + c\sqrt{\frac{r}{\lambda}}, \quad (9)$$

where $c = 0.20 \pm 0.01$.

These results allow for a statement on the limiting case of infinite Reynolds numbers, $\mathrm{Re} \to \infty$. According to Eq. (8), the coefficients α and δ tend to zero [18]. $\gamma(r)$ does not depend on Re. Thus drift and diffusion coefficients take the following simple form for $\mathrm{Re} \to \infty$:

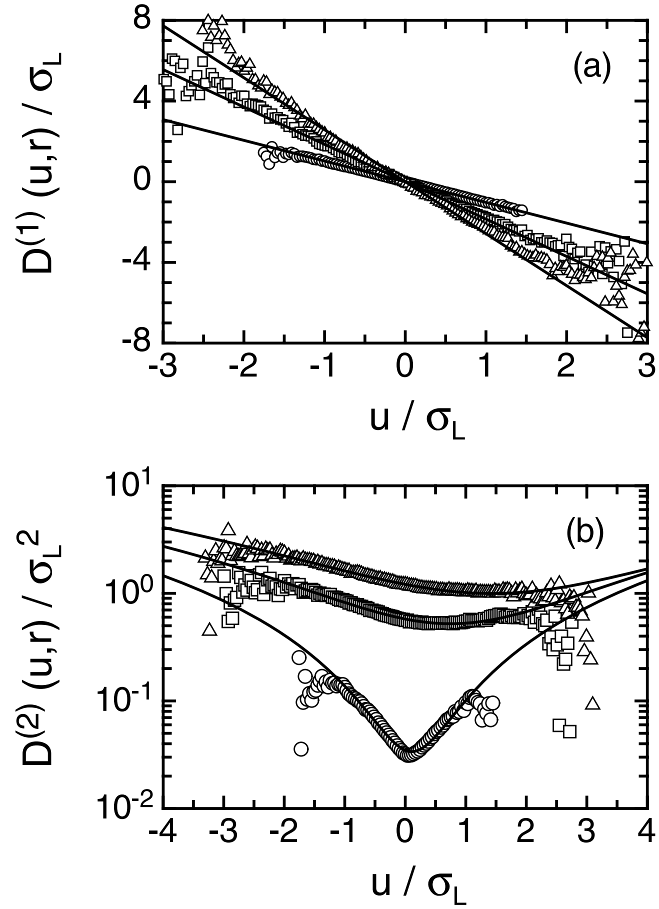

FIG. 2. Coefficients $D^{(1)}(u, r)$ (a) and $D^{(2)}(u, r)$ (b) as functions of the velocity increment u at $r = 3\lambda$ (circles), $r = L/2$ (squares), and $r = L$ (triangles). The dotted curves correspond to linear (a) and polynomial (b) (degree two) fits to the measured data.

$$D_\infty^{(1)}(u, r) = -\gamma(r)u, \qquad D_\infty^{(2)}(u, r) = \beta_\infty(r)u^2. \quad (10)$$

Based on this limiting result, we discuss implications for the structure functions $S_u^n(r)$. This should quantify the importance of the coefficients α and δ and, thus, give evidence how far away our measured data are from a universal state, which should show scaling behavior. After the multiplication of the corresponding Fokker-Planck equation (3) for $p(u, r)$ with u^n from left and successively integrating with respect to u, the equation

$$\frac{r\frac{\partial}{\partial r}S_u^n(r)}{nS_u^n(r)} = \gamma(r) - (n-1)\beta_\infty(r) \quad (11)$$

is obtained.

According to Kolmogorov's four-fifth law (cf. [1]), the third order structure function, $S_u^3(r)$, is proportional to r. Thus, for $n = 3$, the left side of Eq. (11) is equal to $1/3$ and $\beta_\infty(r)$ is given by:

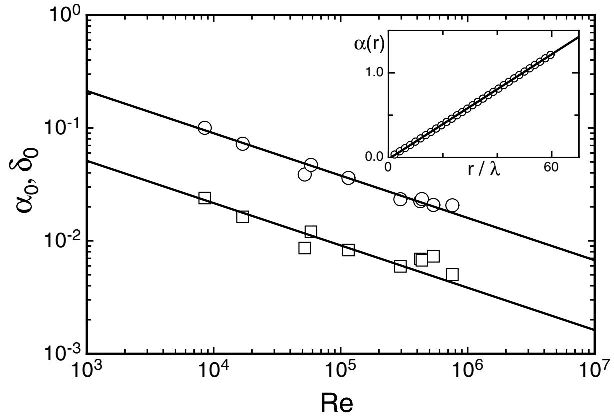

FIG. 3. Coefficients α_0 (circles) and δ_0 (squares) defined in Eqs. (6) and (7) as functions of the Reynolds number Re; lines represent power laws in Re with a scaling exponent of $-3/8$. The inset displays $\alpha(r)$ as a function of the length scale r for $R_\lambda = 1180$. The best fit to the data point give exponents of -0.36 ± 0.05.

$$\beta_\infty(r) = \frac{\gamma(r)}{2} - \frac{1}{6}. \quad (12)$$

For increasing Re, the experimental results for $\beta(r)$ in fact show a tendency towards the limiting value β_∞ (see Fig. 4), but it is also clearly observed that the convergence is slow and that even the highest accessible Re numbers are still far from this limiting case. This is in accordance with recent theoretical findings [19].

To summarize, the framework of Markov processes can successfully be applied to characterize the stochastic behavior of turbulence with increasing Re number. Moreover, this description is complete in the sense that the entire information about any N-scale pdf $p(u_1, r_1; u_2, r_2; \ldots; u_N, r_N)$, is encoded in the two coefficients $D^{(1)}(u, r)$ and $D^{(2)}(u, r)$. We find rather simple dependencies on their arguments u, r, and the Re number.

The Re dependence of the coefficients, especially of $D^{(2)}$, yields strong experimental evidence for a significant change of the stochastic process as the Re number increases. This finding clearly contradicts the concept of a universal turbulent cascade and might also be of importance in large eddy simulations where the influence of the subgrid stress on the large scale dynamics of a turbulent flow is modeled under the assumption of universality.

It is easily verified that, according to Eq. (11), the increase of $\beta(r)$ with Re excludes the simple scaling laws proposed by Kolmogorov in 1941 [3]. The universal functional dependence of $\gamma(r)$ on r [Eq. (9)] does not support the value of $\gamma \approx 1/3$ [10,20]. The obvious dependence of the coefficients γ and β on r also contradicts that structure functions exhibit proper scaling behavior for all orders n, as can be derived from Eq. (11). This does not say that tendencies to commonly known scaling behavior are present.

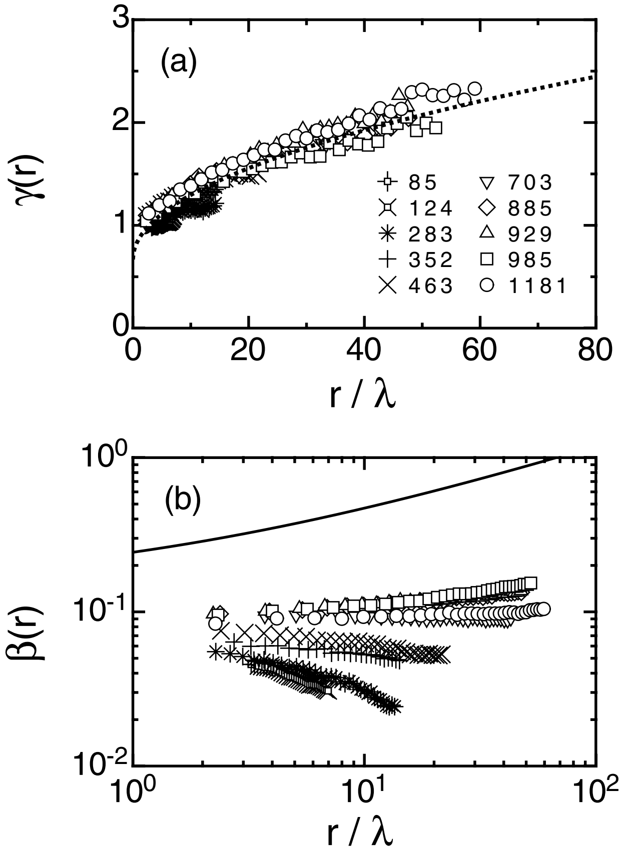

FIG. 4. The slope $\gamma(r)$ of $D^{(1)}$ (a) and the quadratic coefficient $\beta(r)$ of $D^{(2)}$ (b) as functions of the scale r for several Reynolds numbers (given by R_λ; see legend). γ is close to a universal function of the scale $\rho = r/\lambda$ [the dotted line is a fit according to Eq. (9)], the coefficient β exhibits a strong dependence on the Reynolds number [17] with a clear tendency towards the limiting value $\beta_\infty(r)$ given by Eq. (12) (full line).

With the limiting values for the coefficients $D^{(k)}$ as given by Eq. (10), the stochastic process for infinite Reynolds numbers corresponds to an infinitely divisible multiplicative cascade [21] as proposed in Ref. [22]. From the slow convergence of the measured coefficient $\beta(r)$ towards its limiting value $\beta_\infty(r)$, it is obvious that turbulent data measured in typical laboratory experiments are still far from that special case. It is therefore important to develop a better understanding of the limiting case $\mathrm{Re} \to \infty$ as indicated, for example, in [23]. It seems questionable whether models on turbulence established under the assumption of infinite Reynolds numbers can be tested in real-life experimental situations at all.

We gratefully acknowledge fruitful discussions with J.-F. Pinton, B. Castaing, F. Chilla, and M. Siefert and support by DFG.

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[16] Note that this method differs slightly from the one proposed in Ref. [11], where first the $M^{(k)}$ were fitted as functions of the velocity increment u. The coefficients of those fits were extrapolated towards $\Delta r = 0$ in a second step.
[17] Error bars as derived in [11] are typically within some percent; thus, the decrease of β values for $R_\lambda = 1181$ are within the error bars. Note that the distance to the limiting case of Eq. (12) is several hundred percent.
[18] Strictly speaking, only the coefficients α_0 and δ_0 tend to zero. But as the ratio L/λ scales like R_λ, the coefficients α and δ at the integral length scale L like $\alpha(L) = \alpha_0 \frac{L}{\lambda} \propto \mathrm{Re}^{-3/8} R_\lambda \propto R_\lambda^{-3/4} R_\lambda \propto R_\lambda^{1/4}$. On the other hand, the coefficient γ scales like the square root of r/λ. Thus, $\gamma(L) \propto \sqrt{L/\lambda} \propto R_\lambda^{1/2}$. The fact that $\gamma(L)$ grows faster than $\alpha(L)$ and $\delta(L)$ justifies neglecting α and δ in the limit $\mathrm{Re} \to \infty$.
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