Relativistic Hydrodynamic Scaling from the Dynamics of Quantum Field Theory

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Relativistic hydrodynamic scaling or boost invariance is a particularly important hydrodynamic regime, describing collective flows of relativistic many body systems and is used in the interpretation of experiments from high-energy cosmic rays to relativistic heavy-ion collisions. We show evidence for the emergence of hydrodynamic scaling from the dynamics of relativistic quantum field theory. We consider a scalar $\lambda\phi^4$ model in $1 + 1$ dimensions in the Hartree approximation and study the relativistic collisions of two kinks and the decay of a localized high-energy density region. We find that thermodynamic scalar isosurfaces show approximate boost invariance at high-energy densities.

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Hydrodynamics has been used since the work of Landau [1], to describe the properties of high multiplicity final states in high-energy particle collisions. While Landau's motivations dealt with high-energy cosmic rays, there has since been ample evidence from accelerator experiments that hydrodynamic scaling (the longitudinal velocity $v_x = x/t$), which implies flat rapidity distributions, is the correct approximate kinematical constraint for the dynamics of high-energy particle collisions.

These findings imply that hydrodynamic scaling must emerge from the dynamics of quantum field theory, if the latter is to be the correct description of the collective behavior in particle physics models. While the applicability of quantum field theory in these regimes is not in doubt, it has not been demonstrated that hydrodynamic scaling, which implies that the energy density isosurfaces are surfaces of constant $\tau^2 = (t^2 - x^2)$, is achieved at sufficiently high center of mass collision energies.

The importance of hydrodynamic scaling is that it leads to a simple understanding of why the single particle distribution functions of outgoing particles have a plateau when plotted against the particle rapidity variable $y = \frac{1}{2}\ln[(E + p_\parallel)/(E - p_\parallel)]$, where E and $p_\parallel$ refer to the energy and momentum in the direction of the collision of an outgoing particle [2]. From a center of mass perspective, one can understand this in terms of having the energy for particle production deposited in a highly Lorentz contracted region so there is no longitudinal scale in the subsequent flow [1,3]. From a frame independent perspective, developed by Bjorken [4], one understands this through the approximate boost invariance (for modest boosts) of two highly Lorentz contracted colliding nuclei at high energies. Field theory calculations using this kinematic constraint also lead to flat rapidity distributions for outgoing particles [5]. As a result, this simplifying constraint is often used in both hydrodynamic and field theory calculations of particle production. The purpose of this Letter is to justify this approximation (scaling) in a first principles field theory calculation.

Present and future experimental prospects for the study of hydrodynamic scaling in high-energy experiments are tremendous. The relativistic heavy ion collider (RHIC) is presently producing the highest energy, highest multiplicity hadronic final states ever achieved in a controlled environment [6]. The large hadron collider (LHC) will later produce even more spectacular events. The detailed understanding of hydrodynamic flows in these experiments constitutes the most promising path for the determination of the thermodynamic properties of nuclear matter at high temperatures [7], viz. its equation of state and the nature of the confinement and chiral symmetry breaking transition.

Direct field theoretical methods, although still in their adolescence [8], offer much promise for the understanding of hydrodynamic scaling and the limits of its applicability. Moreover, they also make accessible more general situations where fields may be strongly out of thermal equilibrium (e.g., at a "quench") or where quantum coherence matters, which escape Boltzmann particle methods.

In this Letter we show, for the first time, how hydrodynamic scaling emerges from the dynamics of a simple $1 + 1$-dimensional scalar field theory in the Hartree approximation. Our results allow us to map the energy density and pressure as a function of space and time. To exhibit the ubiquity of hydrodynamic scaling, we study two different situations: one in which a hot region is formed in the wake of the collision of two leading particles (kinks) at relativistic velocities, and another simpler one where we construct a local energy overdensity which is allowed to relax under its own self-consistent evolution. To be definite, we will be concerned with a scalar $\lambda\phi^4$ quantum field theory with Lagrangian density

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\mu_\Lambda^2\phi^2 - \frac{1}{4}\lambda\phi^4. \quad (1)$$

The well-known Hartree approximation is the simplest nontrivial truncation of the coupled equations for the field's correlation functions, which assumes that all *connected* correlation functions beyond the second are negligible [9]. This leads to a dynamical equation for the mean field $\varphi \equiv \langle \phi \rangle$ and the connected two-point function. We write the quantum field $\phi = \varphi + \hat{\psi}$, where $\hat{\psi}$ are fluctuations, $\langle \hat{\psi} \rangle = 0$. The equations of motion for φ and the two-point function $G(x, y) = \langle \hat{\psi}(x)\hat{\psi}(y) \rangle$ then are

$$\begin{aligned} &[\Box - \mu_\Lambda^2 + \lambda\varphi^2(x) + 3\lambda G(x,x)]\varphi(x) = 0, \\ &[\Box - \mu_\Lambda^2 + 3\lambda(\varphi^2(x) + G(x,x))]G(x,y) = 0. \end{aligned} \tag{2}$$

To solve the equation for the Green's functions, we will rely on a complete orthogonal mode basis $\psi_k(x)$

$$\hat{\psi}(x) = \sum_k [a_k^\dagger \psi_k^*(x) + a_k \psi_k(x)], \tag{3}$$

where $a_k^\dagger$, a_k are creation and annihilation operators obeying canonical commutation relations. In terms of the mode fields ψ_k, at zero temperature

$$G(x, y) = \sum_k \psi_k(x)\psi_k^*(y). \tag{4}$$

The effective mass squared of the propagator $\chi(x, t)$ must be finite, which tells us how to choose the bare mass μ_Λ^2. In 1 + 1 dimensions the self-energy has only a logarithmic divergence, which is eliminated by a simple mass renormalization. We choose

$$-\mu_\Lambda^2 = \pm m^2 - 3\lambda \int \frac{dk}{2\pi} \frac{1}{2\sqrt{k^2 + \chi}} \equiv \pm m^2 - 3\lambda G_0, \tag{5}$$

leading to the existence of two homogeneous stable phases, corresponding to $\chi_0 = m^2$, $\phi = 0$ and $\chi_0 = 2m^2$, $\phi^2 = m^2/\lambda$, i.e., a symmetric and a broken symmetry phase, respectively. The renormalized equations are

$$\begin{aligned} &[\Box + \chi(x) - 2\lambda\varphi^2(x)]\varphi(x) = 0, \\ &[\Box + \chi(x)]\psi_k(x, t) = 0 \qquad \forall_k, \end{aligned} \tag{6}$$

$$\chi(x) = \pm m^2 + 3\lambda\varphi^2(x) + 3\lambda G_R(x, x), \tag{7}$$

where the renormalized $G_R(x, x) = G(x, x) - G_0$. The challenge posed by Eqs. (6) and (7) in spatially inhomogeneous cases is that we need to solve many partial differential equations simultaneously. In a spatial lattice of linear size L, the computational effort is of order L^{2D}, per time step, where D is the number of space dimensions. Because of this demanding scaling, we focus on $D = 1$.

We consider two classes of initial conditions: (i) colliding kinks [10] in the broken phase where, at $t = 0$, we have

$$\varphi(x, t = 0) = \frac{m}{\sqrt{\lambda}} \varphi_{\text{kink}}(x - x_0)\varphi_{\text{kink}}(-x - x_0), \tag{8}$$

$$\varphi_{\text{kink}}(x) = \tanh(mx/\sqrt{2}), \tag{9}$$

with the kinks initially boosted towards each other at velocity v, and (ii) a Gaussian shape in the unbroken phase

$$\varphi(x, 0) = \varphi_0 \exp\left[-\frac{x^2}{2A}\right], \qquad \partial_t\varphi(x, 0) = 0. \tag{10}$$

In both cases we adopt at $t = 0$ a Fourier plane-wave mode basis, characteristic of the unperturbed vacuum

$$\psi_k(x, t) = \sqrt{\frac{\hbar}{2\omega_k}} e^{i(kx + \omega_k t)}, \qquad \omega_k = \sqrt{k^2 + \chi_0}. \tag{11}$$

The orthonormality of the basis is preserved by the evolution, Eqs. (6) and (7).

To study the hydrodynamic behavior, we need to specify the operator energy momentum tensor $T^{\mu\nu}$

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}L. \tag{12}$$

Its expectation value, in terms of φ and ψ_k, is

$$\begin{aligned} \langle T_{00} \rangle &= \frac{1}{2}(\varphi_t)^2 + \frac{1}{2}(\varphi_x)^2 + \frac{1}{2}\sum_k [|\psi_t^k|^2 + |\psi_x^k|^2] + V_{\mathrm{H}}, \\ \langle T_{11} \rangle &= \frac{1}{2}(\varphi_t)^2 + \frac{1}{2}(\varphi_x)^2 + \frac{1}{2}\sum_k [|\psi_t^k|^2 + |\psi_x^k|^2] - V_{\mathrm{H}}, \\ V_{\mathrm{H}} &= \frac{\chi^2}{12\lambda} - \lambda\frac{\varphi^4}{2}, \\ \langle T_{01} \rangle &= \langle T_{10} \rangle = \varphi_t\varphi_x + \frac{1}{2}\sum_k [\psi_x^k\psi_t^{*k} + \psi_t^k\psi_x^{*k}], \end{aligned} \tag{13}$$

where all arguments are at x. The subscripts x, t are shorthand for spatial and time derivatives, respectively. T_{00} and T_{11} contain two different types of ultraviolet divergent contributions. The first arises from the one-loop integral $G(x, x)$. This logarithmic divergence is removed by mass renormalization Eq. (7). The second divergence appears in the kinetic and spatial derivative fluctuation terms. This divergence is purely quadratic and is already present in the free field theory in the vacuum sector. By comparing the mode sum with a covariant dimensional regularization scheme for the free field theory, one deduces that the correct subtraction in the mode sum scheme is given by

$$\frac{1}{2}\sum_k [|\psi_t^k|^2 + |\psi_x^k|^2] \to \frac{1}{2}\sum_k [|\psi_t^k|^2 + |\psi_x^k|^2 - |k|]. \tag{14}$$

In practice, we discretize the fields $\varphi(x)$ and the set $\{\psi_j\}$ on a spatial lattice with size N and spacing dx and use periodic boundary conditions in space. We choose $dx = 0.125$, $N = 1024$, and $m^2 = 1$, $\lambda = 1$, $\hbar = 1$. The dynamical equations are solved using a symplectic fourth order integrator (with a time step $dt = 0.025$). With these choices, in the finite volume $L = Ndx$, the momentum k takes a finite number of discrete values $k_n = \frac{2\pi n}{L}$, with $n = \{-\frac{N}{2}, \ldots, \frac{N}{2} - 1\}$ and continuum k integrals become sums $\int dk/(2\pi) \to L^{-1}\sum_n$. The frequency ω_k now satisfies a lattice form of the dispersion relation, with

$$\omega_k^2 = \hat{k}^2 + \chi^2, \qquad \hat{k}^2 = \frac{2}{dx^2}(1 - \cos dxk_n). \tag{15}$$

These forms also require that the renormalization of $T_{\mu\nu}$ be achieved using appropriate lattice choices. In particular, we adopt $|k_n| = \sqrt{\hat{k}^2}$ in (14).

We are now ready to address the hydrodynamics of our field theory. In $1 + 1$ dimensions, one can diagonalize the expectation value of the energy momentum tensor and cast in the form of an ideal fluid [11]

$$\langle T^{\mu\nu}\rangle = (\varepsilon + p)u^\mu u^\nu - g^{\mu\nu}p; \qquad \partial^\mu\langle T_{\mu\nu}\rangle = 0, \tag{16}$$

where $u^\mu = \gamma(1, v)$, $(\gamma = 1/\sqrt{1 - v^2})$ is the collective fluid velocity, and ε and p are the comoving energy and pressure densities. The latter are the eigenvalues of the energy momentum tensor and can be obtained from the invariance of its trace and determinant $\varepsilon - p = T^\mu_\mu$, $\varepsilon p = \mathrm{Det}|T|$. The fluid velocity can be obtained from (16)

$$T^{01} = (\varepsilon + p)\frac{v}{1 - v^2}. \tag{17}$$

The attraction of scaling lies in the fact that the relation $x = \pm vt$ allows for great simplifications of the hydrodynamic equations (16), which can then be expressed in terms of a single variable and thus become ordinary differential equations. These can then be solved analytically [3], generating predictions for the spatiotemporal behavior of hydrodynamic quantities such as energy density ε, temperature, or entropy. The important feature of these solutions, in 1D flow, is that boost invariance requires that they are functions of proper time τ only and are independent of rapidity. This feature is preserved by the hydrodynamic evolution. To be explicit, we consider a simple equation of state $dp/d\varepsilon = c_0^2$, with c_0 the (constant) speed of sound. This describes, in particular, the ultrarelativistic free gas with $c_0 = 1$. Then [3]

$$\varepsilon(x, t)/\varepsilon_0 = (\tau/x_0)^{-(1+c_0^2)}; \qquad \tau = \sqrt{t^2 - x^2}, \tag{18}$$

where ε_0, x_0 are integration constants. The equation of state $p = c_0^2\varepsilon$ can be obtained using simple assumptions about the (hadronic) excitation spectrum [12,13]. The dependence of thermodynamic scalars on τ alone implies that their isosurfaces are hyperboloids $t^2 - x^2 = \mathrm{const}$ in space-time, a property that we can easily check in our results, see Figs. 1 and 3 corresponding to initial conditions (i) and (ii), respectively. Figure 2 shows the pressure in space-time for the same situation as in Fig. 1.

Scaling solutions do not preserve global energy conservation. Thus, energy isosurfaces must eventually deviate from the scaling hyperbola and join neighboring isosurfaces, creating characteristic hornlike shapes. This behavior, which can be extracted directly from hydrodynamic equations, is also observed in the quantum field solutions (Figs. 1 and 3).

Equation (18) also allows us a measurement of c_0. Figure 4 shows the decay in time of the energy density of the Gaussian mean field discussed in Fig. 3. The

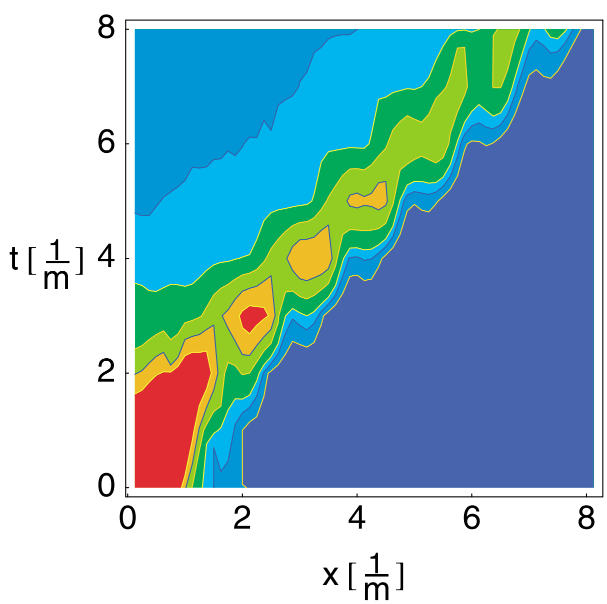

FIG. 1 (color). Contours of equal energy density in space-time, near the collision point of two kinks (at the origin of the plot), initially boosted towards each other at $v = 0.8$. Contours are for $\epsilon/m^2 = 0.1, 0.08, 0.06, 0.04, 0.03, 0.02$. The collision is symmetric under spatial reflection. We show the region after and to the right of the collision point. Red denotes the highest energy density, navy blue the lowest. The energy isosurfaces show signs of hydrodynamic scaling, following approximate hyperboloids, which are distorted because of the presence of the emerging kinks.

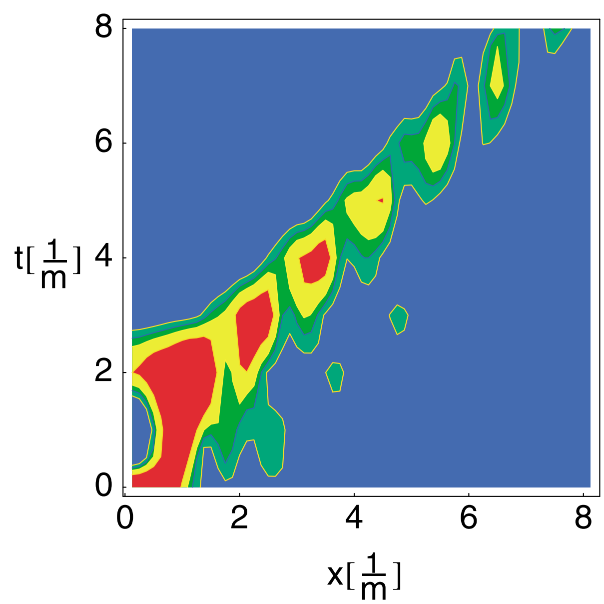

FIG. 2 (color). Pressure isosurfaces in space-time for the situation shown in Fig. 1. Asymptotically far from the kink trajectories, the equation of state mimics that of a gas with $p = c_0^2\varepsilon$. Close to the kinks the pressure gradients indicate space-time regions where strong energy flows are imminent, such as the area around the kink collision point, at the origin of the plot.

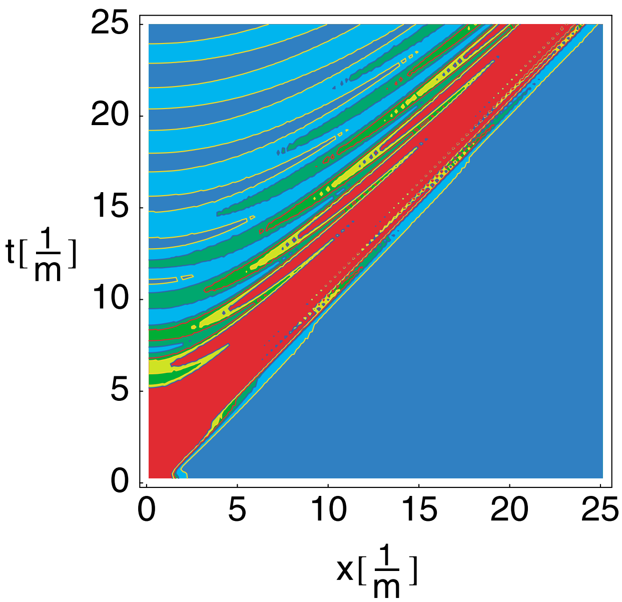

FIG. 3 (color). ε isosurfaces in space-time, for the decay of an initial Gaussian shape (10), with $A = 1$ and $\varphi_0 = 5$. Contours are for $\epsilon/m^2 = 1, 0.9, 0.7, 0.5, 0.3, 0.1$. Here we chose the energy contained in the initial hot region to be about 2 orders of magnitude larger than that deposited by the kink collision of Figs. 1 and 2. The red region carries away most of the energy in the form of a wave packet traveling close to the speed of light. The energy isosurfaces follow exquisite hyperboloids, characteristic of relativistic hydrodynamic scaling.

resulting fit suggests a value of $1 \geq c_0 \gtrsim 0.77$, compatible with an ultrarelativistic equation of state $c_0 = 1$ in 1D.

In conclusion, we have demonstrated for the first time that hydrodynamic scaling emerges from the dynamics of quantum field theory at sufficiently high-energy densities. We analyzed situations both with leading particles, the asymptotic states both before and after the collision, and the evolution of a simple local energy overdensity. The extension of this type of calculation to 3D, where hydrodynamics is richer, and to include scattering, necessary for the description of real fluids, remain necessary steps to make real time studies of quantum fields predictive experimentally in the context of heavy-ion collisions.

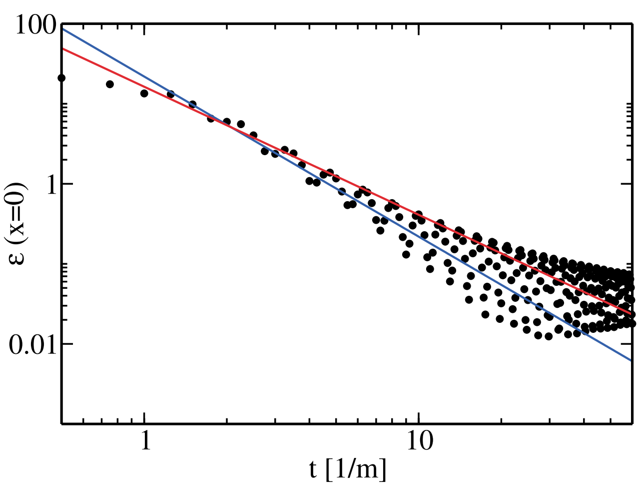

FIG. 4 (color). $\varepsilon(x = 0)/m^2$, in the wake of the decay of a Gaussian hot region of Fig. 3. The solid lines show power laws of the form (18) with $1 + c_0^2 = 2$ (blue), corresponding to the ultrarelativistic ideal gas ($c_0 = 1$) and with $1 + c_0^2 = 1.6$ (red).

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