Two- and Three-Dimensional Oscillons in Nonlinear Faraday Resonance

I. V. Barashenkov, N. V. Alexeeva, and E. V. Zemlyanaya

Department of Maths and Applied Maths, University of Cape Town, Rondebosch 7701, South Africa

(Received 1 June 2001; published 15 August 2002)

We study 2D and 3D localized oscillating patterns in a simple model system exhibiting nonlinear Faraday resonance. The corresponding amplitude equation is shown to have exact soliton solutions which are found to be always unstable in 3D. On the contrary, the 2D solitons are shown to be stable in a certain parameter range; hence the damping and parametric driving are capable of suppressing the nonlinear blowup and dispersive decay of solitons in two dimensions. The negative feedback loop occurs via the enslaving of the soliton's phase, coupled to the driver, to its amplitude and width.

DOI: 10.1103/PhysRevLett.89.104101 PACS numbers: 05.45.Yv, 45.70.Qj, 47.54.+r

Oscillons are localized two-dimensional oscillating structures which have recently been detected in experiments on vertically vibrated layers of granular materials [1], Newtonian fluids, and suspensions [2,3]. Numerical simulations established the existence of stable oscillons in a variety of pattern-forming systems, including the Swift-Hohenberg and Ginsburg-Landau equations, period-doubling maps with continuous spatial coupling, semicontinuum theories, and hydrodynamic models [3,4]. Although these simulations provided a great deal of insight into the phenomenology of the oscillons (in particular, demarcated their existence area on the corresponding phase diagrams), little is known about the mechanism by which they acquire or loose their stability.

In this Letter, we consider a generic model of nonlinear parametric resonance in a distributed system—which has *exact* oscillon solutions and allows an accurate characterization of their existence and stability domains. The main purpose of this work is to understand how the oscillons manage to resist the nonlinear blowup and dispersive decay which are characteristic for localized excitations in 2D media. Our model admits a straightforward generalization to three dimensions, and we use this opportunity to explore the existence of stable oscillons in 3D as well.

The model consists of a D-dimensional lattice of parametrically driven nonlinear oscillators (e.g., pendula) [5] with the nearest-neighbor coupling:

$$\frac{d^2}{d\tau^2}\phi_{\mathbf{k}} + \alpha \frac{d}{d\tau}\phi_{\mathbf{k}} + 2\kappa D\,\phi_{\mathbf{k}} - \kappa \sum_{|\mathbf{m}-\mathbf{k}|=1} \phi_{\mathbf{m}} + (1+\rho\cos 2\omega\tau)\sin\phi_{\mathbf{k}} = 0, \tag{1}$$

where $\mathbf{k} = (k_1, \dots, k_D)$. Assuming that the coupling is strong, $\kappa \gg 1$; that the damping and driving are weak, $\alpha = \gamma\varepsilon^2$, $\rho = 2h\varepsilon^2$, where $\varepsilon \ll 1$; and that the driving half-frequency is just below the edge of the linear spectrum gap, $\omega^2 = 1 - \varepsilon^2$, the oscillators execute small-amplitude librations of the form $\phi_{\mathbf{k}} = 2\varepsilon\psi(t, \mathbf{x}_{\mathbf{k}})e^{-i\omega\tau} + \text{c.c.} + O(\varepsilon^3)$, where $t = \varepsilon^2\tau/2$, $\mathbf{x}_{\mathbf{k}} = (\varepsilon/\sqrt{\kappa})\mathbf{k}$, and the slowly varying amplitude satisfies

$$i\psi_t + \nabla^2\psi + 2|\psi|^2\psi - \psi = h\psi^* - i\gamma\psi, \tag{2}$$

the parametrically driven damped nonlinear Schrödinger (NLS) equation. Although our lattice model is not aimed at the accurate description of any particular experimental situation, we note that Eq. (2) does have specific applications, e.g., describes an optical resonator with different losses for the two polarization components of the field [6]. It was also used as a phenomenological model of nonlinear Faraday resonance in water [3].

In the absence of the damping and driving, all localized initial conditions in the 2D and 3D NLS equations are known either to disperse or blow up in finite time [7–9]. Surprisingly, numerical simulations of Eq. (2) with sufficiently large h and γ revealed the occurrence of stable (or possibly long-lived) stationary localized excitations [3]. No analytic solutions were found, however, and a possible stabilization mechanism remained unclear [10].

In fact, there are two exact (though not explicit) stationary radially symmetric solutions given by

$$\psi^{\pm} = \mathcal{A}_{\pm}e^{-i\theta_{\pm}}\,\mathcal{R}_0(\mathcal{A}_{\pm}r), \tag{3}$$

where $\mathcal{A}_{\pm}^2 = 1 \pm \sqrt{h^2-\gamma^2}$, $\theta_+ = \frac{1}{2}\arcsin(\gamma/h)$, $\theta_- = \frac{\pi}{2} - \theta_+$, and $\mathcal{R}_0(r)$ is the bell-shaped nodeless solution of

$$\nabla_r^2\mathcal{R} - \mathcal{R} + 2\mathcal{R}^3 = 0; \qquad \mathcal{R}_r(0) = \mathcal{R}(\infty) = 0, \tag{4}$$

with $\nabla_r^2 = \partial_r^2 + \frac{D-1}{r}\partial_r$. (Below we simply write $\mathcal{R}$ for $\mathcal{R}_0$.) Solutions of Eq. (4) in $D = 2$ and 3 are well documented in literature [7]. One advantage of having an explicit dependence on h and γ is that the existence domain is characterized by an explicit formula. The soliton ψ^+ exists for all $h > \gamma$ and the ψ^- for $\gamma < h < \sqrt{1+\gamma^2}$. It is pertinent to add here that for $h < \gamma$, all initial conditions decay to zero. This follows from the rate equation

$$\partial_t|\psi|^2 = 2\nabla(|\psi|^2\nabla\chi) + 2|\psi|^2(h\sin 2\chi - \gamma), \tag{5}$$

where $\psi = |\psi|e^{-i\chi}$. Defining $N = \int|\psi|^2 d\mathbf{x}$, Eq. (5) implies $N_t \le 2(h-\gamma)N$ whence $N(t) \to 0$ as $t \to \infty$.

We now examine the stability of the two solitons. Linearizing Eq. (2) in the small perturbation

 0031-9007/02/89(10)/104101(4)$20.00

$$\delta\psi(\mathbf{x},t) = e^{(\mu-\Gamma)\tilde{t}-i\theta_\pm}[u(\tilde{\mathbf{x}}) + iv(\tilde{\mathbf{x}})], \quad (6)$$

where $\tilde{\mathbf{x}} = \mathcal{A}_\pm \mathbf{x}$, $\tilde{t} = \mathcal{A}_\pm^2 t$, we get an eigenproblem

$$L_1 u = -(\mu+\Gamma)v, \qquad (L_0 - \epsilon)v = (\mu - \Gamma)u, \quad (7)$$

where $\Gamma = \gamma/\mathcal{A}_\pm^2$ and the operators

$$L_0 \equiv -\tilde{\nabla}^2 + 1 - 2\mathcal{R}^2(\tilde{r}), \qquad L_1 \equiv L_0 - 4\mathcal{R}^2(\tilde{r}), \quad (8)$$

with $\tilde{\nabla}^2 = \sum_{i=1}^D \partial^2/\partial\tilde{x}_i^2$. (We are dropping the tildas below.) The quantity ϵ, $\epsilon = \pm 2\sqrt{h^2-\gamma^2}/\mathcal{A}_\pm^2$, is positive for the ψ^+ soliton and negative for ψ^-. Each ϵ defines a "parabola" on the (h,γ) plane:

$$h = \sqrt{\epsilon^2/(2-\epsilon)^2 + \gamma^2}. \quad (9)$$

Introducing $\lambda^2 = \mu^2 - \Gamma^2$ and changing $v(\mathbf{x}) \to (\mu+\Gamma)\lambda^{-1}v(\mathbf{x})$ [11], Eq. (7) is reduced to a *one*-parameter eigenvalue problem:

$$(L_0 - \epsilon)v = \lambda u, \qquad L_1 u = -\lambda v. \quad (10)$$

Since $\mathcal{R}_0(r)$ is nodeless in $0 \le r < \infty$, and $L_0\mathcal{R}_0 = 0$, the operator $L_0 - \epsilon$ is positive definite for $\epsilon < 0$. In this case the eigenvalue can be found as a minimum of the Rayleigh quotient:

$$-\lambda^2 = \min_w \frac{\langle w|L_1|w\rangle}{\langle w|(L_0-\epsilon)^{-1}|w\rangle}. \quad (11)$$

The operator L_1 has D zero eigenvalues associated with the translation eigenfunctions $\partial_i\mathcal{R}(r)$, $i = 1, 2, \ldots, D$; hence it also has a negative eigenvalue with a radial-symmetric eigenfunction $w_0(r)$. Substituting w_0 into the quotient in (11), we get $-\lambda^2 < 0$ whence $\mu > \Gamma$. Thus the soliton ψ^- is unstable (against a nonoscillatory mode) for all D, h, and γ and may be safely disregarded.

Before proceeding to the stability of ψ^+ (for which we have $\epsilon > 0$), we make a remark on the undamped, undriven case ($\epsilon = 0$). In three dimensions, the eigenvalue problem (10) has a zero eigenvalue associated with the phase invariance of the unperturbed NLS equation (2) and another one, associated with the scaling symmetry:

$$\begin{pmatrix} L_0 & 0 \\ 0 & L_1 \end{pmatrix}\begin{pmatrix} \mathcal{R} \\ -\frac{1}{2}(r\mathcal{R})_r \end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{R} \end{pmatrix}. \quad (12)$$

Both the eigenvector $(\mathcal{R},0)^T$ and the rank-2 generalized eigenvector $(0, -\frac{1}{2}(r\mathcal{R})_r)^T$ are radially symmetric. In 2D the number of repeated zero eigenvalues associated with radially symmetric invariances is four; in addition to those in (12) we have a two-parameter group of the lens transformations [7,8] giving rise to

$$\begin{pmatrix} L_0 & 0 \\ 0 & L_1 \end{pmatrix}\begin{pmatrix} \frac{1}{8}r^2\mathcal{R} \\ g \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(r\mathcal{R})_r \\ \frac{1}{8}r^2\mathcal{R} \end{pmatrix}, \quad (13)$$

with some $g(r)$. When $h^2 - \gamma^2$ (or, equivalently, ϵ) deviates from zero, all the above invariances break down and the two (respectively, four) eigenvalues move away from the origin on the plane of complex λ. The directions of their motion are crucial for the stability properties.

We can calculate $\lambda(\epsilon)$ perturbatively, assuming

$$\begin{aligned} \lambda &= \lambda_1\epsilon^{1/4} + \lambda_2\epsilon^{2/4} + \lambda_3\epsilon^{3/4} + \ldots, \\ u &= u_1\epsilon^{1/4} + u_2\epsilon^{2/4} + \ldots, \\ v &= \mathcal{R} + v_1\epsilon^{1/4} + v_2\epsilon^{2/4} + \ldots, \end{aligned} \quad (14)$$

where $v_i = v_i(r)$, $u_i = u_i(r)$. Substituting into (10), the order $\epsilon^{1/4}$ gives $u_1 = -\lambda_1 L_1^{-1}\mathcal{R}$. Using (12), u_1 is found explicitly: $u_1 = (\lambda_1/2)(r\mathcal{R})_r$. At the order $\epsilon^{2/4}$ we get $u_2 = -\lambda_2 L_1^{-1}\mathcal{R}$ and equation $L_0 v_2 = \lambda_1 u_1$. Since L_0 has a null eigenvector, $\mathcal{R}(r)$, it is solvable only if

$$\lambda_1\int \mathcal{R}(r)u_1(r)d\mathbf{x} = -\lambda_1^2\frac{D-2}{4}\int\mathcal{R}^2(r)d\mathbf{x} = 0. \quad (15)$$

In the two-dimensional case the condition (15) is satisfied for any λ_1, whereas in $D = 3$ we have to set $\lambda_1 = 0$. Next, at the orders $\epsilon^{3/4}$ and $\epsilon^{4/4}$ we obtain, respectively,

$$L_0 v_3 = \lambda_2 u_1 + \lambda_1 u_2 = \lambda_1\lambda_2(r\mathcal{R})_r, \quad (16)$$

$$L_0 v_4 = R + \lambda_1 u_3 + \lambda_2 u_2 + \lambda_3 u_1. \quad (17)$$

Equation (16) is solvable both in 2D and 3D. The solvability condition for (17) reduces to

$$\lambda_1^4 = -\frac{\langle\mathcal{R}|\mathcal{R}\rangle}{\langle\mathcal{R}|L_1^{-1}L_0^{-1}L_1^{-1}|\mathcal{R}\rangle} = -16\frac{\int\mathcal{R}^2 d\mathbf{x}}{\int\mathcal{R}^2 r^2 d\mathbf{x}}, \quad (18)$$

$$\lambda_2^2 = \frac{\langle\mathcal{R}|\mathcal{R}\rangle}{\langle\mathcal{R}|L_1^{-1}|\mathcal{R}\rangle} = 4, \quad (19)$$

in two and three dimensions, respectively.

Thus we arrive at two different bifurcation scenarios. In 3D, where $\lambda_1 = 0$ and λ_2 is real, two imaginary eigenvalues $\pm|\lambda_2|\epsilon^{1/2}$ converge at the origin as $\epsilon \to 0$ from the left. (This does not mean that the ψ^- soliton is stable as there still is a pair of finite real eigenvalues for $\epsilon < 0$.) As ϵ grows to positive values, the imaginary pair $\pm|\lambda_2|\epsilon^{1/2}$ moves onto the real axis. A numerical study [12] of the eigenvalue problem (10) shows that when ϵ is further increased, the four real eigenvalues collide, pairwise, and acquire imaginary parts. Importantly, for all $0 < \epsilon < 1$ the imaginary parts remain smaller in magnitude than the real parts. This means that $\mathrm{Re}\mu$ remains greater than Γ all the time, implying that the three-dimensional ψ^+ soliton is unstable for all h and γ.

The bifurcation occurring in 2D is more unusual. As ϵ approaches zero from the left, *four* eigenvalues converge at the origin, two along the real and two along the imaginary axis: $\lambda \approx \pm|\lambda_1|(-\epsilon)^{1/4}, \pm i|\lambda_1|(-\epsilon)^{1/4}$. As ϵ moves to positive, the four eigenvalues start diverging at 45° to the real and imaginary axes. Hence to the leading order, $\mathrm{Im}\lambda \approx \mathrm{Re}\lambda$, and in order to make a conclusion about the stability, we need to calculate the higher-order corrections. The order $\epsilon^{5/4}$ produces a solvability condition

$$\lambda_1^3\lambda_2\langle\mathcal{R}|L_1^{-1}L_0^{-1}L_1^{-1}|\mathcal{R}\rangle = \frac{\lambda_1^3\lambda_2}{16}\int \mathcal{R}^2 r^2 d\mathbf{x} = 0,$$

yielding $\lambda_2 = 0$. [Here we made use of (13).] Finally, the order $\epsilon^{6/4}$ defines λ_3 [where $g(r)$ is as in (13)]:

$$\lambda_3 = \frac{1}{\lambda_1} + \frac{\lambda_1^3}{2}\frac{\int g(r)\mathcal{R}(r)r^2 d\mathbf{x}}{\int \mathcal{R}^2(r)r^2 d\mathbf{x}}. \qquad (20)$$

Taking λ_1 in the first quadrant, $\lambda_1 = e^{i\pi/4}|\lambda_1|$, and doing the integrals in (18) and (20) numerically, we conclude that λ_3 is in the second quadrant, $\lambda_3 = e^{3i\pi/4}|\lambda_3|$, which implies that $|\mathrm{Im}\lambda| > |\mathrm{Re}\lambda|$. In terms of λ, the stability criterion $\mathrm{Re}\mu \le \Gamma$ is written as $\gamma \ge \gamma_c$, where

$$\gamma_c(\epsilon) \equiv \frac{2}{2-\epsilon}\frac{\mathrm{Re}\lambda(\epsilon)\,\mathrm{Im}\lambda(\epsilon)}{\sqrt{(\mathrm{Im}\lambda)^2 - (\mathrm{Re}\lambda)^2}}. \qquad (21)$$

The smallest γ for which the soliton can be stable equals

$$\lim_{\epsilon\to 0}\gamma_c(\epsilon) = \frac{1}{2\sqrt{2}}|\lambda_1|^{3/2}|\lambda_3|^{-1/2}. \qquad (22)$$

Substituting for λ_1, λ_3 their numerical values, (22) gives $\gamma_c(0) = 1.006\,47$. For $\epsilon \neq 0$ we obtained $\lambda(\epsilon)$ by solving the eigenvalue problem (10) directly [12]. Here we have restricted ourselves to radially symmetric $u(r)$ and $v(r)$. Expressing ϵ via γ_c from (21) and feeding into (9), we get the stability boundary on the (h, γ) plane (Fig. 1).

Asymmetric perturbations do not lead to any instabilities in 2D. To show this, we factorize, in (10), $u(\mathbf{x}) = \tilde{u}(r)e^{im\varphi}$ and $v(\mathbf{x}) = \tilde{v}(r)e^{im\varphi}$, where $\tan\varphi = y/x$ and m is an integer. The eigenproblem (10) remains the same, with only the operators L_0 and L_1 being replaced by

$$L_0^{(m)} \equiv L_0 + m^2/r^2, \qquad L_1^{(m)} \equiv L_1 + m^2/r^2. \qquad (23)$$

The number of the discrete eigenvalues of $L_0^{(m)}$ is limited by the Bargmann bound [13]: $\mathrm{n}^{(m)} \le \frac{1}{m}\int \mathcal{R}^2 r dr$. Numerically, $\int \mathcal{R}^2 r dr = 0.93$; hence the spectrum of $L_0^{(m)}$ is purely continuous (and extends from 1 to ∞). Therefore the operator $L_0^{(m)} - \epsilon$ with $\epsilon < 1$ is positive definite, and the eigenvalues of the problem (10) can be found from the variational principle (11). The operator $L_1^{(1)}$ has a zero eigenvalue with the eigenfunction $w^{(1)}(r) = \mathcal{R}_r(r)$ which has no nodes for $0 < r < \infty$; hence all its other eigenvalues (if any exist) are positive. This also implies that operators $L_1^{(m)}$ with $m^2 > 1$ are positive definite, as $L_1^{(m)} - L_1^{(1)} > 0$. Thus the minimum of the quotient (11) is zero for $m^2 = 1$ and positive for $m^2 > 1$, and so all λ^2 are ≤ 0.

Besides the nodeless solution $\mathcal{R}_0(r)$, the "master" equation (4) has solutions $\mathcal{R}_n(r)$ with n nodes, $n \ge 1$. These give rise to a sequence of nodal solutions of the damped-driven NLS (2), defined by Eq. (3) with $\mathcal{R}_0 \to \mathcal{R}_n$. It is easy to realize that the solutions ψ_n^- are unstable against radially symmetric nonoscillatory modes for all h, γ, n, and D. (The proof is a simple generalization of the one for ψ_0^-.) To examine the stability of the ψ_n^+, we solved the eigenvalue problem (10) numerically, with operators $L_{0,1}^{(m)}$ as in (23). In 3D, positive real eigenvalues (with radially symmetric eigenfunctions) are present in the spectrum for all ϵ; thus the three-dimensional nodal solutions are always prone to a symmetric collapse or dispersive spreading. In 2D, the ψ_n^+ solutions are stable against radially symmetric perturbations for sufficiently large γ but turn out to be always unstable against azimuthal perturbations. In particular, the ψ_1^+ solution has instabilities associated with $1 \le m \le 5$, and the $m = 4$ mode has the largest growth rate for all ϵ. The corresponding eigenvalue λ is real and the eigenfunctions $u(r)$ and $v(r)$ have a single maximum near the position of the lateral minimum of the function $\mathcal{R}_1(r)$. Following Ref. [14] where a similar scenario was described for nodal optical waveguides, the above observation suggests that the ψ_1^+ solution will break into a symmetric pattern of five solitons ψ_0^+: one at the origin and four others around it, standing still or drifting away with exponentially small velocities. Next, the ψ_2^+ solution has azimuthal instabilities with $1 \le m \le 10$. The analysis of the corresponding eigenfunctions suggests that, depending on h and γ, the breakup products will comprise a necklace of 11 to 13 ψ_0^+ solitons: one at the origin, three or four around it, and seven or eight forming an outer ring. We verified these predictions via direct numerical simulations of the time-dependent array (1); the simulations corroborated the above scenario. Thus the nodal solutions can be interpreted as degenerate, unstable coaxial complexes of the nodeless solitons.

Last, we need to understand the stabilization mechanism in qualitative terms. Equation (2) is derivable from the stationary action principle with the Lagrangian

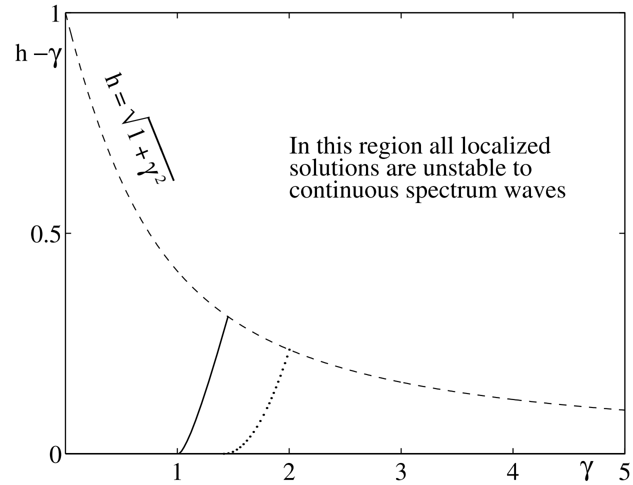

FIG. 1. Stability diagram for two-dimensional solitons. The $(\gamma, h-\gamma)$ plane is used for visual clarity. No localized or periodic attractors exist for $h < \gamma$ (below the horizontal axis). The region of stability of the soliton ψ^+ lies to the right of the solid curve. The dotted curve gives the variational approximation to the stability boundary of the ψ^+ soliton: $h = (1+\gamma^4)^{1/2}$, $\gamma \ge \sqrt{2}$.

$$\mathcal{L} = e^{2\gamma t}\mathrm{Re}\int(i\psi_t\psi^* - |\nabla\psi|^2 - |\psi|^2 + |\psi|^4 - h\psi^2)d\mathbf{x}.$$

Choosing the ansatz $\psi = \sqrt{A}e^{-i\theta-(B+i\sigma)r^2}$ [15,16] with A, B, θ, σ functions of t, this reduces, in 2D, to

$$\mathcal{L} = e^{2\gamma t}\frac{A}{B}\bigg[\dot\theta - 1 + \frac{\dot\sigma}{2B} - \frac{2B}{\cos^2\phi} + \frac{A}{2} - h\cos(\phi + 2\theta)\cos\phi\bigg], \qquad (24)$$

where $\tan\phi = \sigma/B$. The four-dimensional dynamical system defined by (24) has two stationary points representing the $\psi^\pm$ solitons. In agreement with the stability properties of the solitons in the full PDE, the ψ^+ stationary point is unstable for small γ but stabilizes for larger dampings (Fig. 1). When γ is large we can expand $A = A_0 + \frac{1}{\gamma}A_1 + \ldots$, $B = B_0 + \frac{1}{\gamma}B_1 + \ldots$, $\theta = \frac{\pi}{4} + \frac{1}{\gamma}\theta_1 + \ldots$, $\sigma = \frac{1}{\gamma}\sigma_1 + \ldots$. Letting $h = \gamma + \frac{c}{2\gamma}$, where $0 \le c \le 1$, defining $T = \frac{t}{\gamma}$, and matching coefficients of like powers of $\frac{1}{\gamma}$ yields a two-dimensional system

$$dA_0/dT = A_0[c + 8\sigma_1 - 4\theta_1^2 + 2(\sigma_1/B_0)^2], \qquad (25)$$

$$dB_0/dT = 8\sigma_1 B_0 + 4\sigma_1\theta_1 + 4(\sigma_1^2/B_0), \qquad (26)$$

$$\theta_1 = \frac{1}{2} + 2B_0 - \frac{3}{4}A_0, \qquad \sigma_1 = \frac{1}{2}A_0B_0 - 2B_0^2. \qquad (27)$$

Like their parent system (24), Eqs. (25)–(27) have two fixed points, the saddle at $B_0^- = \frac{1}{2} - \sqrt{c}$, $A_0^- = 4B_0^-$ and a stable focus at $B_0^+ = \frac{1}{2} + \sqrt{c}$, $A_0^+ = 4B_0^+$.

According to (5), the soliton's phase $\chi = \theta + \sigma r^2$ controls the creation and annihilation of the soliton's elementary constituents (whose density is $|\psi|^2$). [If Eq. (2) is used to model Faraday resonance in fluids [3], $\int|\psi|^2 d\mathbf{x}$ has the meaning of the mass of the fluid captured in the oscillon.] Since the creation and annihilation occurs mainly in the core of the soliton [see (5)], the variable phase component σr^2 plays a marginal role in this process. Instead, the significance of the quantity σ is in that it controls the flux of the constituents between the core and the periphery of the soliton—see the χ_r term on the right-hand side of (5).

If we perturb the stationary point ψ^+ in the four-dimensional phase space of (24), the variables θ and σ will zap, within a very short time $\Delta t \sim \frac{1}{\gamma}$, onto the two-dimensional subspace defined by the constraints (27). After this short transient the evolution of θ and σ will be immediately following that of the soliton's amplitude $\sqrt{A}$ and width $1/\sqrt{B}$. In the case of the ψ^+ soliton, this provides negative feedback: perturbations in A and B produce only such changes in the phase and flux that the new values of θ and σ stimulate the recovery of the stationary values of A and B. (The phase θ works to restore the number of constituents while σ rearranges them within the soliton.) In the case of ψ^- the feedback is positive: the perturbation-induced phase and flux (27) strive to amplify the perturbation of the soliton's amplitude and width still further. Finally, for small γ the coupling of θ and σ to A_0 and B_0 is via differential rather than algebraic equations. In this case the dynamics of the phase and flux is inertial and their changes may not catch up with those of the amplitude and width. The feedback loop is destroyed and the soliton destabilizes.

We thank D. Astruc, S. Flach, S. Longhi, and D. Pelinovsky for useful discussions. The work of E. Z. was supported by RFBR Grant No. 0001-00617.

[1] P. B. Umbanhowar, F. Melo, and H. L. Swinney, Nature (London) **382**, 793 (1996).
[2] O. Lioubashevski *et al.*, Phys. Rev. Lett. **83**, 3190 (1999); H. Arbell and J. Fineberg, *ibid.* **85**, 756 (2000).
[3] D. Astruc and S. Fauve, in *Fluid Mechanics and Its Applications* (Kluwer, Dordrecht, 2001), Vol. 62, pp. 39–46.
[4] L. S. Tsimring and I. S. Aranson, Phys. Rev. Lett. **79**, 213 (1997); E. Cerda, F. Melo, and S. Rica, *ibid.* **79**, 4570 (1997); S. C. Venkataramani and E. Ott, *ibid.* **80**, 3495 (1998); D. Rothman, Phys. Rev. E **57**, 1239 (1998); J. Eggers and H. Riecke, *ibid.* **59**, 4476 (1999); C. Crawford and H. Riecke, Physica (Amsterdam) **129D**, 83 (1999); H. Sakaguchi and H. R. Brand, Europhys. Lett. **38**, 341 (1997).
[5] Y. Braiman, J. F. Lindner, and W. L. Ditto, Nature (London) **378**, 465 (1995); M. Weiss, T. Kottos, and T. Geisel, Phys. Rev. E **63**, 056211 (2001).
[6] V. J. Sánchez-Morcillo *et al.*, Opt. Lett. **25**, 957 (2000).
[7] K. Rypdal, J. J. Rasmussen, and K. Thomsen, Physica (Amsterdam) **16D**, 339 (1985), and references therein.
[8] E. A. Kuznetsov and S. K. Turitsyn, Phys. Lett. **112A**, 273 (1985); V. M. Malkin and E. G. Shapiro, Physica (Amsterdam) **53D**, 25 (1991).
[9] For a recent review and references on the blowup in 2D and 3D NLS equations, see, e.g., L. Berge, Phys. Rep. **303**, 259 (1998); G. Fibich and G. Papanicolaou, SIAM J. Appl. Math. **60**, 183 (1999); P. M. Lushnikov and M. Saffman, Phys. Rev. E **62**, 5793 (2000).
[10] In the discrete model (1), the blowup will of course be arrested when the collapsing spike becomes localized on one or a few sites. Our aim here is to demonstrate that a combination of damping and driving can suppress the blowup long before that, while the NLS approximation is still valid.
[11] I. V. Barashenkov, M. M. Bogdan, and V. I. Korobov, Europhys. Lett. **15**, 113 (1991).
[12] For the solution of Eq. (4) we used a fourth-order Newtonian iteration, on the intervals (0,30) and (0,45), with the stepsize $\Delta r = 2.5 \times 10^{-3}$ and 1.25×10^{-3}. The eigenproblem (10) was solved by the Fourier expansion over 200 modes (and verified with 400 and 800 modes).
[13] R. G. Newton, J. Operator Theory **10**, 119 (1983).
[14] J. M. Soto-Crespo *et al.*, Phys. Rev. A **44**, 636 (1991).
[15] This ansatz is frequently used in studies of blowup phenomena; see, e.g., V. E. Zakharov and E. A. Kuznetsov, Sov. Phys. JETP **64**, 773 (1986).
[16] S. Longhi, G. Steinmeyer, and W. S. Wong, J. Opt. Soc. Am. B **14**, 2167 (1997).