

## ***N*-Particle *N*-Level Singlet States: Some Properties and Applications**

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Three apparently unrelated problems which have no solution using classical tools are described: the “*N*-strangers,” “secret sharing,” and “liar detection” problems. A solution for each of them is proposed. Common to all three solutions is the use of quantum states of total spin zero of *N* spin- $(N - 1)/2$  particles.

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Not long ago, during a meeting on quantum information, a speaker asked the participants to make a list of “interesting” quantum states, namely, those which have potential applications (particularly tasks which were impossible using classical physics) or illustrate fundamental issues of quantum mechanics [1]. The final list was rather short: Einstein-Podolsky-Rosen-Bohm-Bell states of two particles [2–4], Greenberger-Horne-Zeilinger (GHZ) states of three or more qubits [5], Werner (mixed) states [6], Hardy states (pure nonmaximally entangled states of two particles) [7], Horodecki “bound” states (entangled mixed states from which no pure entanglement can be distilled) [8], and *W* states of three qubits [9]. Curiously, most of them were first introduced in connection with Bell’s theorem, and some appeared in the course of the classification of entanglement. Surprisingly, none of them was originally introduced as the answer to a practical problem without classical solution (although most of them have later found numerous applications [10–19]).

Here we shall introduce three apparently unrelated problems without classical solution and then propose a solution for all of them, using quantum mechanics. Common to all these solutions is the use of a family of quantum states.

(1) *The *N* strangers problem.*—The scenario for this problem is an extension to a high number *N* of players of the situation described in Patricia Highsmith’s novel and Alfred Hitchcock’s movie *Strangers on a Train* [20]: *N* complete strangers  $A_i$  ( $i = 1, \dots, N$ ), meet on a train.  $A_i$  wants  $B_i$  dead. During small talk, one suggests that an “exchange” murder between *N* complete strangers would be unsolvable. After all, how could the police find the murderer when he/she is a total and complete stranger with absolutely no connection whatsoever to the murdered victim?  $A_i$  could kill  $B_k$ , etc. [21]. However, such a plan suffers from an important problem: if all the players know who the murderer of each victim is, then the whole plan is vulnerable to individual denunciations. Alternatively, if the distribution of victims is the result of a secret lottery, how could the murderers be assured that the lottery was not rigged and that nobody had contrived the result or could ascertain it?

The problem is then how to distribute the victims  $\{B_i\}_1^N$  among the murderers  $\{A_i\}_1^N$ , which share no previous secret information nor any secure classical channel, in a way that

guarantees that each murderer  $A_i$  knows only the identity of his/her victim and that nobody else (beside the murderers) knows anything about the assignment of the victims.

(2) *The secret sharing problem.*—This problem was already described, for  $N = 3$ , in [15]. It could arise in the following context:  $A_1$  wants to have a secret action taken on her behalf at a distant location. There she has  $N - 1$  agents,  $A_2, A_3, \dots, A_N$  who carry it out for her.  $A_1$  knows that some of them are dishonest, but she does not know which one it is. She cannot simply send a secure message to all of them, because the dishonest ones will try to sabotage the action, but it is assumed (as in [15]) that if all of them carry it out together, the honest ones will keep the dishonest ones from doing any damage.

The problem is then that  $A_1$  wishes to convey a cryptographic key to  $A_2, A_3, \dots, A_N$  in such a way that none of them can read it on their own, only if all the  $A_i$  ( $i = 2, 3, \dots, N$ ) collaborate. In addition, they wish to prevent any eavesdropper from acquiring information without being detected. It is assumed that  $A_1$  shares no previous secret information nor any secure classical channel with her agents.

Different quantum solutions to this problem for  $N = 3$  have been proposed using either GHZ [14,15] or Bell states [16]. Below we shall propose a different solution for any *N* which exhibits some additional advantages.

(3) *The liar detection problem.*—Let us consider the following scenario: three parties *A*, *B*, and *C* are connected by secure pairwise classical channels. Let us suppose that *A* sends a message *m* to *B* and *C*, and *B* sends the same message to *C*. If both *A* and *B* are honest, then *C* should receive the same *m* from *A* and *B*. However, *A* could be dishonest and send different messages to *B* and *C*,  $m_{AB} \neq m_{AC}$  (Fig. 1, left), or, alternatively, *B* could be dishonest and send a message which is different to the one he receives,  $m_{BC} \neq m_{AB}$  (Fig. 1, right). For *C* the problem is to ascertain without any doubt who is being dishonest. This problem is interesting for classical information distribution in pairwise connected networks. The message could be a database and the dishonest behavior a consequence of an error during the copying or distribution process. This problem has no solution by classical means. It is at the heart of a slightly more complicated problem in distributed computing called the Byzantine agreement problem [22], a version

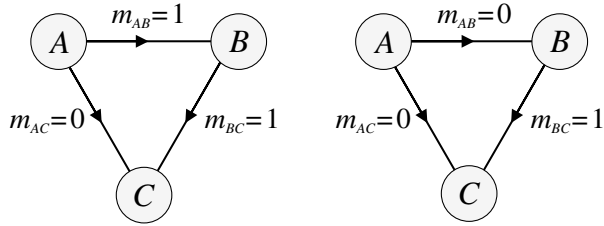


FIG. 1. The liar detection problem. Left: A is a liar because she sends different messages to B and C. Right: B is a liar because he sends C a message different from the one he received from A. The task is for C to identify who is being dishonest, A or B.

of which has been recently solved using quantum means by Fitz, Gisin, and Maurer [23]. Indeed, the solution for our liar detection problem is based on theirs.

The next step is to show that all of these problems can be solved if each of the  $N$  participants are in possession of a sequence of numbers with the following properties: (i) It is *random* (i.e., generated by an intrinsically unrepeatable method which gives each possible number with the same probability of occurrence). (ii) The possible numbers are integers from 0 to  $N - 1$ . (iii) If a number  $i$  is at position  $j$  of the sequence of party  $k$ ,  $i$  is not at position  $j$  in the sequence of a different party. (iv) Each party knows only his/her own sequence. (v) Nobody else (beside the parties) knows the sequences. Properties (iv) and (v) are difficult to accomplish using classical tools due to the fact that information transmitted in classical form can be examined and copied without altering it in any detectable way. However, as quantum key distribution protocols show [10,24], quantum information does not suffer from such a drawback.

Assuming we have a reliable method to generate sequences of numbers with properties (i) to (v) among  $N$  distant parties, a method that will be presented below, then the solutions to the above problems are as follows.

*Solution to the  $N$  strangers problem.*—Each victim  $B_i$  is assigned a label, taken from 0 to  $N - 1$ . If murderer  $A_i$ 's sequence starts with  $j$ , then  $A_i$  must kill  $B_j$ , etc. The remaining entries of the sequence can be used for subsequent rounds. The result tells every murderer who his/her victim is in such a way that prevents any murderer (or even a small group of them) from denouncing or blackmailing another. The only way to ascertain with certainty who murdered  $B_j$  is that all the other murderers confess [25].

*Solution to the secret sharing problem.*—The key is defined as  $A_1$ 's sequence. The only way to reveal it is to make the remaining  $N - 1$  parties share their respective sequences; the key is then composed by the missing results. If a dishonest party  $D$  declares a result which is different to his/her actual result, then there is a probability  $1/(r - 1)$ , where  $r$  is the number of honest parties which have not yet declared their results, that other honest party  $H$  has obtained that result. Then  $H$  would stop the process, so Alice's key (and thus Alice's action) would remain safe

(dishonest parties cannot sabotage Alice's action if they do not know what it is). The order in which the agents declare their respective results must change from round to round to avoid any dishonest party being always the last to declare.

*Solution to the liar detection problem.*—Let us suppose that the message  $m$  is a trit value 0, 1, or 2. The three parties agree to use the following protocol: (I) If the transmitted message is  $m_{ij}$ , then the sender  $i$  must also send  $j$  the list  $l_{ij}^{(m_{ij})}$  of positions in his/her sequence in which the number  $m_{ij}$  appears. Note that if the sequences are random and long enough then any  $l_{ij}^{(m_{ij})}$  must contain about one third of the total length  $L$  of the sequences. (II) The receiver  $j$  would not accept any message if the intersection between the received list  $l_{ij}^{(m_{ij})}$  and his/her list  $l_j^{(m_{ij})}$  is not null nor if  $l_{ij}^{(m_{ij})} \ll L/3$  elements. We will assume that requirements (I) and (II) force the dishonest one to send correct but perhaps incomplete lists. Otherwise, if  $i$  sends a list containing  $n$  erroneous data, then the probability that  $j$  does not accept the message  $m_{ij}$  would be  $(2^n - 1)/2^n$ . In addition, (III)  $B$  must send  $C$  the list  $l_{BC}^{(m_{AB})}$  containing the sequence he has (supposedly) received from A. Therefore, when  $C$  finds that  $m_{AC} \neq m_{BC}$ , she has received three lists to help her to find out whether it is A or B who is being dishonest.

According to rules (I) to (III), if  $B$  wants to be dishonest  $l_{BC}^{(m_{BC})} \cup l_{BC}^{(m_{AB})}$  must necessarily be a subset of  $l_B^{(m_{BC})}$ , because  $B$  does not know  $l_A^{(m_{BC})}$ . However, the length of  $l_B^{(m_{BC})}$  is about  $L/3$ , while  $C$  is expecting  $B$  to send her two lists with a total length of  $2L/3$ ; then  $C$  would conclude that  $B$  was being dishonest. Alternatively, if it is A who is being dishonest, the lengths of the two lists that  $C$  received from  $B$  would total about  $2L/3$ ; then  $C$  would conclude that A was being dishonest.

The next step is to present a method to generate among  $N$  distant parties sequences of numbers with properties (i) to (iv). A possible quantum solution, probably not the only one, but maybe the most natural, is by distributing among all  $N$  parties an  $N$ -particle  $N$ -level singlet state of total spin zero. For arbitrary  $N$  these states can be expressed as

$$|S_N\rangle = \frac{1}{\sqrt{N!}} \sum_{\substack{\text{permutations} \\ \text{of } 01\dots(N-1)}} (-1)^t |ij\dots n\rangle, \quad (1)$$

where  $t$  is the number of transpositions of pairs of elements that must be composed to place the elements in canonical order (i.e., 0, 1, 2, ...,  $N - 1$ ) and  $|01\dots(N-1)\rangle$  denotes the tensor product state  $|0\rangle \otimes |1\rangle \otimes \dots \otimes |N-1\rangle$ . Particular examples of  $|S_N\rangle$  are as follows:

$$|S_2\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle), \quad (2)$$

$$|S_3\rangle = \frac{1}{\sqrt{6}}(|012\rangle - |021\rangle - |102\rangle + |120\rangle + |201\rangle - |210\rangle), \quad (3)$$

$$|S_4\rangle = \frac{1}{\sqrt{24}}(|0123\rangle - |0132\rangle - |0213\rangle + |0231\rangle + |0312\rangle - |0321\rangle - \dots + |3210\rangle). \quad (4)$$

If we identify subsystems with spin- $(N-1)/2$  particles and associate the state  $|s\rangle$  with the eigenvalue  $s - (N-1)/2$  of the spin observable in some fixed direction, then  $|S_N\rangle$  is the only state of  $N$  particles of spin- $(N-1)/2$  which has total spin zero.

The described solutions assume that the  $N$  parties share a large collection of  $N$ -level systems in the  $|S_N\rangle$  state. This requires a protocol to distribute and test these states between the  $N$  parties such that at the end of the protocol either all parties agree that they share a  $|S_N\rangle$  state (and then they can reliably apply the described solutions), or all of them conclude that something went wrong (and then abort any subsequent action). For  $N=3$  such a distribute-and-test protocol is explicitly described in [23] and can be easily be generalized to any  $N > 3$ . The test requires that the parties compare a sufficiently large subset of their particles which are subsequently discarded.

For our purposes, some interesting properties of the states  $|S_N\rangle$  are as follows:

(a) They provide *correlated* results. As can be easily seen in Eq. (1), whenever the  $N$  parties measure the spin of the  $N$  separated particles along the direction used in Eq. (1), each of them finds a different result in the set  $\{0, \dots, N-1\}$ ; thus such results satisfy requirements (ii) and (iii).

(b) Moreover,  $|S_N\rangle$  are  *$N$ -lateral rotationally invariant*. This means that if we act on any of them with the tensor product of the  $N$  rotation operators referring to all the particles for any arbitrary rotation, the result will be to reproduce the same state (within a possible phase factor). Therefore, whenever the  $N$  parties measure the spin of the  $N$  separated particles *along any direction* (but it must be the same direction for everyone), each of them finds a different result in the set  $\{0, \dots, N-1\}$ ; thus such results satisfy requirements (ii) and (iii). Therefore, the direction of measurement could be randomly chosen and publicly announced (once the particles have been distributed among the parties) before any set of measurements.

(c) In order to accomplish (i), (iv), and (v), an essential property is *nonseparability*, that is, the quantum predictions for the states  $|S_N\rangle$  cannot be reproduced by any local hidden variables model in which the results of the spin measurements are somehow determined before the measurement. To show the nonseparability of  $|S_N\rangle$  we have to study whether they violate Bell's inequalities derived from the assumptions of local realism. Most Bell's inequalities require two alternative local *dichotomic* (taking values  $-1$  or  $1$ ) observables  $A_j$  and  $B_j$  on each particle  $j$ . To test

nonseparability, we will use the dichotomic local observables proposed by Peres in [26]. A Peres' observable  $A_k$  can be operationally defined as follows: to measure  $A_k$ , first measure the spin component of particle  $k$  along direction  $A$ ,  $S_A^{(k)}$ . If particle  $k$  is a spin- $s$  particle, then measuring  $S_A^{(k)}$  could give  $2s+1$  different results. Then assign value  $1$  to results  $s, s-2$ , etc., and value  $-1$  to results  $s-1, s-3$ , etc. The operator representing observable  $A_k$  can be written as

$$\hat{A}_k = \sum_{m=-s}^s (-1)^{s-m} |S_A^{(k)} = m\rangle \langle S_A^{(k)} = m|, \quad (5)$$

where  $|S_A^{(k)} = m\rangle$  is the eigenstate of the spin component along direction  $A$  of particle  $k$ .

Probably the simplest way to show the nonseparability [27] of the  $|S_N\rangle$  states is by considering the following scenario:  $N$  distant observers share  $N$   $N$ -level particles in the  $|S_N\rangle$  state; the  $N-m$  observers can choose between measuring  $A_j = A$  and  $B_j = a$ ; the remaining  $m$  observers can choose between measuring  $A_k = B$  and  $B_k = b$ . Then nonseparability can be tested by means of the following Bell's inequality, which generalizes to  $N$  particles the Clauser-Horne-Shimony-Holt (CHSH) inequality [28]

$$|E_N(A, \dots, A, B, \dots, B) + E_N(A, \dots, A, b, \dots, b) + E_N(a, \dots, a, B, \dots, B) - E_N(a, \dots, a, b, \dots, b)| \leq 2. \quad (6)$$

Note that this inequality uses only a subset of all possible correlation functions [for instance, it does not use  $E_N(A, a, \dots, a, B, \dots, B)$ ]. Restricting our attention to Peres' observables, for the states  $|S_N\rangle$ , the correlation function  $E_N^{(m)}(A, \dots, A, B, \dots, B)$ , which represents the expectation value of this product of the results of measuring, for instance,  $N-m$  observables  $A$ , and  $m$  observables  $B$  is given by

$$E_N^{(N-1)} = (-1)^{f(N/2)} \frac{1}{N} \frac{\sin(N\theta_{AB})}{\sin\theta_{AB}}, \quad (7)$$

$$E_N^{(N-2)} = (-1)^{f(N/2)} \frac{1}{N+2} \left[ 1 + \frac{\sin[(N+1)\theta_{AB}]}{\sin\theta_{AB}} \right], \quad (8)$$

where  $\theta_{AB}$  is the angle between directions  $A$  and  $B$  and  $f(x)$  gives the greatest integer less than or equal to  $x$ . In case of  $m=1$ , that is, using correlation functions of the  $E_N^{(N-1)}$  type, we have found that states  $|S_n\rangle$  violate inequality (6) for any  $N$ . The maximum violation for  $N=2$  is  $2\sqrt{2}$ , for  $N=3$  is  $2.552$ , and for  $N \rightarrow \infty$  tends to  $2.481$ . In case of  $m=2$ , that is, using correlation functions of the  $E_N^{(N-2)}$  type, we have found that the states  $|S_n\rangle$  violate inequality (6) for any  $N$ . The maximum violation for  $N=4$  is  $2.418$ , for  $N=5$  is  $2.424$  and for  $N \rightarrow \infty$  tends to  $2.481$ .

(d) Nonseparability is robust against the loss of any number of parties if they can publicly announce the results of their measurements. For instance, let us suppose that  $N-2$  observers measure the spin along the same direction

A and publicly announce their results. Then, if the missing results are  $j$  and  $k$ , the state shared by the remaining two observers is

$$|\sigma\rangle_N = \frac{1}{\sqrt{2}}(|S_A = j, S_A = k\rangle_N - |S_A = k, S_A = j\rangle_N). \quad (9)$$

Note that  $|\sigma\rangle_N$  is formally similar to the singlet state of two qubits. However, it belongs to the  $N^2$ -dimensional Hilbert space  $\mathcal{H}_N \otimes \mathcal{H}_N$  and not to  $\mathcal{H}_2 \otimes \mathcal{H}_2$ , and therefore does not exhibit rotational symmetry. For Peres' observables, the corresponding correlation function is

$$E_N^\sigma(A, B) = (-1)^N \cos^{N-1} \theta_{AB}. \quad (10)$$

The states  $|\sigma\rangle_N$  violate the CHSH inequality. For  $N = 2$  the maximum violation is  $2\sqrt{2}$ , for  $N = 3$  the maximum violation is 2.414, and for  $N \rightarrow \infty$  tends to 2.324.

A similar situation occurs when any number  $p$  of observers (not necessarily two) measure the same spin component and publicly announce their results. Then, the state shared by the remaining  $N - p$  observers is formally similar to  $|S_{N-p}\rangle$ , but belongs to a  $N^2$ -dimensional Hilbert space  $\mathcal{H}_N \otimes \mathcal{H}_N$ . Therefore, in the secret sharing scenario, if some of the parties get caught by the enemy, and they are nevertheless able to publicly announce their results, the remaining parties still share pseudo  $|S_{N-p}\rangle$  states and could still use them for secret sharing.

Only very recently has it been possible to prepare optical analogs to the singlet states of two  $N$ -level systems for every  $N$  [29] and to test Bell inequalities for two qutrits [30]. So far, of the states  $|S_N\rangle$ , only the simplest one, the singlet state of two qubits, has been created in a laboratory. Preparing these states for  $N \geq 3$  is a formidable physical challenge. The aim of this Letter has been to point out some potential applications of these states in order to stimulate the interest in that challenge.

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