Hiding Classical Data in Multipartite Quantum States

T. Eggeling* and R. F. Werner†

Institut für Mathematische Physik, TU Braunschweig, Mendelssohnstrasse 3, 38106 Braunschweig, Germany

(Received 6 March 2002; published 13 August 2002)

We present a general technique for hiding a classical bit in multipartite quantum states. The hidden bit, encoded in the choice of one of two possible density operators, cannot be recovered by local operations and classical communication without quantum communication. The scheme remains secure if quantum communication is allowed between certain partners, and can be designed for any choice of quantum communication patterns to be secure, but to allow near perfect recovery for all other patterns. No entanglement is needed since the hiding states can be chosen to be separable. A single ebit of prior entanglement is not sufficient to break the scheme.

DOI: 10.1103/PhysRevLett.89.097905 PACS numbers: 03.67.–a, 03.65.Ud

INTRODUCTION

Many secrets in the world are locked away with keys distributed among many parties. A well-known classical scheme for this is Shamir's secret sharing [1], in which a preassigned fraction of the key-possessing parties needs to contribute their parts of the key to unlock the secrets. There are two directions in which this can be generalized to hiding information in multipartite quantum states. In the first version called "quantum secret sharing," the bit is hidden in a way that some parties can recover the bit via local operations and classical communication [2,3]. Typically the information is then hidden in pure states and the theory is closely related to the theory of error correcting codes, the errors corresponding to the parties whose part of the key is not available. In the second version, which has been called "quantum data hiding" [4,5], and which we follow in this paper, one still hides a classical bit, but the quantum structure is used to increase the demands on the communication needed for the recovery. Arbitrary classical communication between N parties (along with arbitrary local quantum operations) is allowed, but only with a preassigned amount of quantum information exchange can the hidden information be retrieved.

In [4,5] only the case $N = 2$ was considered. Since the hiding states have very high symmetry in that case (they are special "Werner states" [6]), Terhal *et al.* suggest that multipartite (i.e., $N > 2$) data hiding scenarios might be based on highly symmetric multipartite entangled states such as the ones studied in our paper [7].

Building on this idea we will generalize data hiding to an extremely versatile scheme: For N-partite systems one can freely choose for which patterns of quantum communication the hidden bit can be retrieved and for which patterns it remains hidden. The level of security can be chosen arbitrarily high: The maximal probability of unwanted recoveries and probability for erroneous identification using an allowed pattern of quantum communication go to zero like the inverse of the dimension of the Hilbert spaces at each site. Expressed in terms of the number of hiding qubits, this is exponentially good.

Surprisingly, no entanglement is needed for this scheme: The hiding states can be chosen to be separable (this was strongly suggested, but not proved in [4,5]). In keeping with this, the scheme cannot be broken with a finite amount of prior entanglement. For an entanglement based scheme one would expect that hiding a single bit between two parties becomes insecure if one ebit of prior entanglement is available to them. However, we will show that the amount of entanglement needed to break security is instead of the order needed to establish quantum communication by teleportation.

In this Letter we will focus on the main points of the construction and the main ideas of the proof. For brevity, we will give details only for the case of $N = 4$ equivalent parties. Full proofs of the case of general N and parties possibly playing different roles, will appear elsewhere [8].

MAIN RESULT

Throughout we will assume that one classical bit has been encoded in the preparation of a multipartite quantum system, by preparing either a density operator ρ_0 or ρ_1 on the Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$. We imagine the N subsystems to be distributed to widely separated laboratories. The aim of the parties is to find out the value of the hidden bit. For this they are allowed arbitrary classical communication and can perform local quantum operations. In addition, they may have established quantum communication lines between some of the labs, and their success will depend crucially on which quantum lines are available. Since we do not distinguish between good and bad quantum lines, this pattern of allowed quantum communication is encoded in a *partition* $\mathcal{P}$ of the N sites into disjoint subsets: Inside each of the subsets arbitrary quantum communication is allowed, so these sites act like one party, but no quantum communication is possible between sites in different subsets of $\mathcal{P}$. For example, the partition $\mathcal{P} = (\{1, 2\}, \{3, 4\})$ means that sites 1 and 2 can exchange quantum information freely, just like 3 and 4, but between these groups only classical communication is allowed.

Whatever procedure the N parties apply will amount to measuring some "analyzing operator" A, $\mathbf{0} \leq A \leq \mathbb{1}$ such that the probability for guessing the value "1" of the hidden bit on an initial preparation ρ is $\mathrm{tr}[\rho A]$. The locality conditions imply that only certain operators A are *admissible for* $\mathcal{P}$. Of course, the parties will try to make $\mathrm{tr}[\rho_1 A] \approx 1$ and $\mathrm{tr}[\rho_0 A] \approx 0$. We say that for a particular pair of hiding states ρ_1, ρ_0 a partition $\mathcal{P}$ is *hiding with quality* ε_1, if $|\mathrm{tr}[(\rho_1 - \rho_0)A]| \leq \varepsilon_1$ for all admissible analyzing operators A. On the other hand, we say that $\mathcal{P}$ is *revealing with quality* ε_2, if for some admissible A we have $|\mathrm{tr}[(\rho_1 - \rho_0)A]| \geq 1 - \varepsilon_2$.

Whoever is hiding the information does not know in advance what communication pattern will be established. But, as our construction will show, the states ρ_1, ρ_0 can be designed such that, for any choice of $\varepsilon_1, \varepsilon_2 > 0$, every partition is either hiding or revealing with quality ε_1 or ε_2, respectively. The set of hiding partitions can be chosen arbitrarily subject *only* to the trivial constraint that for every partition which is finer than a hiding one, i.e., which corresponds to a pattern allowing less quantum communication, must itself be hiding. We note that the Hilbert space dimensions need to become large if the ε_i are small. In fact, in our construction the ε_i typically behave like $1/d$, if d is the dimension of the one-site Hilbert spaces. The construction naturally also yields *separable* states ρ_1, ρ_0 satisfying the conditions, although for these still higher dimensions d are required to achieve the same errors.

In this Letter we will explicitly construct hiding states ρ_1, ρ_0 for all choices of hiding partitions of four parties, which are democratic in the sense that each site plays the same role. It is remarkable that two such choices are not comparable in the sense that neither allows more communication than the other: We will give states for which any $2:2$ partition $(\{1,2\},\{3,4\})$ is hiding and any $3:1$ partition $(\{1,2,3\},\{4\})$ is revealing, but also states for which the opposite is true. Hence the "hiding strength" of pairs of states cannot be parametrized by a one-dimensional scale.

CONSTRUCTION

Symmetric states.—We begin by restricting ourselves to a class of highly symmetric states known as multipartite Werner states [7]. Their main virtue is that they can be described by a fixed set of parameters while the local Hilbert space dimensions go to infinity. By definition, four-partite Werner states live on $(\mathbb{C}^d)^{\otimes 4}$, and commute with all unitary operators of the form $U^{\otimes 4}$ with U a unitary operator on the d-dimensional Hilbert space $\mathbb{C}^d$. This is equivalent to the possibility of writing the state as linear combinations of permutation operators (see [9]). For any permutation π of the four sites we will denote the corresponding permutation operator by $V_\pi := \sum_{i,j,k,l=1}^{d} |\pi(ijkl)\rangle\langle ijkl|$.

Since the communication patterns we consider are invariant under permutations, we can even choose the states to be permutation symmetric. We denote by $(i_1, i_2, \ldots, i_r)$ the cyclic permutation $i_1 \mapsto i_2 \mapsto \cdots \mapsto i_r \mapsto i_1$. Then we must have, e.g., $\mathrm{tr}[\rho_i V_{(12)}] = \mathrm{tr}[\rho_i V_{(23)}]$, since these permutations differ only by a relabeling of the sites. This leaves just four expectations characterizing the state, namely,

$$\begin{aligned} r_2 &= \mathrm{tr}[\rho_i V_{(12)}], & r_{22} &= \mathrm{tr}[\rho_i V_{(12)(34)}], \\ r_3 &= \mathrm{tr}[\rho_i V_{(123)}], & r_4 &= \mathrm{tr}[\rho_i V_{(1234)}]. \end{aligned} \tag{1}$$

We will fix this vector $\vec{\rho} = (r_2, r_{22}, r_3, r_4)$ of expectations independently of the dimension d. Thus we automatically get hiding schemes, which work for all dimensions, though achieving $\varepsilon_1 \to 0$ only in the limit $d \to \infty$. Whether or not a particular vector of expectations corresponds to a family of density operators can be decided independently of the dimension by group theoretical criteria, the extremal possibilities being given by irreducible representations of the permutation group. For details we refer to [8].

Analyzing operators for fixed $\mathcal{P}$.—Without loss of discriminating power we can then suppose that the analyzing operators A also have the $U^{\otimes 4}$ symmetry: The four parties only have to perform the same random unitary rotation at every site ("twirling") before realizing their procedure. The resulting A will commute with $U^{\otimes 4}$ but will have exactly the same discriminating power for states insensitive to such unitary rotations. Hence we can write

$$A = \sum_\pi a_\pi V_\pi \tag{2}$$

with suitable coefficients a_π. Note that this averaging operation does *not* work for the permutation symmetry, because the permutations are nonlocal operations, which would clearly require the exchange of quantum information.

It turns out that in the sum (2) we must distinguish two types of terms depending on how the permutation π relates to the partition $\mathcal{P}$. We say that π is *adapted* to $\mathcal{P}$ if π maps each of the sets in the partition into itself. Clearly, if only the coefficients a_π for π adapted to $\mathcal{P}$ are nonzero, A is a local operator in this communication situation, hence admissible. Only such local operators will be needed to show that certain patterns are revealing in our theory.

The key problem (settled in the following subsection) is the converse, namely, to show that every operator A which is admissible for the partition $\mathcal{P}$ is at least approximately of this sort. Fortunately, we can use here the same simple criterion already employed in [4,5], which is based on *partial transposition.* The partial transpose operation Θ_S associated with a subset $S \subset \{1,2,3,4\}$ of the sites takes a tensor product operator $A_1 \otimes \cdots \otimes A_4$ to a similar product, in which all A_i with $i \in S$ are replaced by their matrix transpose in a fixed basis. For example, $\Theta_{\{2,3\}}$ transposes only the second and the third tensor factors of the input. The arguments in [4,5] then tell us that, for any operator A, which is admissible for $\mathcal{P}$, we must have that

$$\mathbf{0} \leq \Theta_S(A) \leq \mathbb{1} \tag{3}$$

for all subsets S compatible with $\mathcal{P}$, i.e., for all S which can be written as unions of the disjoint subsets forming the partition $\mathcal{P}$. Since positivity is preserved under global transposition, it suffices to verify this for either S or its complement. For example, for $\mathcal{P} = (\{1\}\{2, 3\}\{4\})$, we must require (3) for the four subsets $S =$ (empty set), $\{1\}$, $\{2, 3\}$, and $\{4\}$.

Coefficients of admissible operators.—In this subsection we sketch the proof of the following Lemma: *Suppose that A is an analyzing operator, which is admissible for the partition $\mathcal{P}$. Then in the sum* (2) *all coefficients a_π with π not adapted to $\mathcal{P}$ are bounded by c/d, where c is a constant depending only on N.*

We will abbreviate by $\mathbf{O}(1/d)$ any terms bounded by a constant times $1/d$, and leave the estimate of the constants to [8]. Consider the matrix M given by $M_{\pi,\sigma} = d^{-4}\mathrm{tr}[V_\pi^* V_\sigma]$. Then since $\mathrm{tr}[V_\pi] = d^c$, where c is the number of cycles in π (including those of length 1), we find $M_{\pi,\sigma} = \delta_{\pi,\sigma} + \mathbf{O}(1/d)$. Thus to leading order in d, the permutation operators are an orthonormal system with respect to the normalized trace. Then by standard perturbation theory the matrix M^{-1} is also close to the identity, and we can approximately determine the coefficients in the sum (2) from

$$a_\pi = d^{-4}\mathrm{tr}[V_\pi^* A] + \mathbf{O}(1/d). \tag{4}$$

A crucial step in our estimate is to get the trace norm ($\|X\|_1 = \mathrm{tr}[\sqrt{X^*X}]$) of partially transposed permutation operators. We claim that

$$\|\Theta_S(V_\pi)\|_1 = d^{4-l_S(\pi)}, \tag{5}$$

where $l_S(\pi)$ denotes the number of points in S, which are mapped outside S. Rather than proving this in general, consider as an example the case $S = \{1, 2\}$ and $\pi = (2, 3)$. Since "1" is fixed and "2" is mapped to "3" outside S, we have $l_S(\pi) = 1$. We can write $\Theta_S(V_\pi) = \Theta_S(\sum_{ijnm} |ijnm\rangle\langle injm|) = \sum_{ijnm} |innm\rangle\langle ijjm|$. This can be written as $d\,\mathbb{1} \otimes P^{(23)} \otimes \mathbb{1}$, where $P^{(23)}$ denotes the one-dimensional projection onto the maximally entangled vector on sites 2 and 3. Thus $\Theta_S(V_\pi)$ has only the nonzero eigenvalue d with multiplicity d^2. This gives $\|\Theta_S(V_\pi)\|_1 = d^3$ as claimed. More generally, $l_S(\pi)$ appears in this computation as the number of repeated indices in either ket or bra in the analogous representation of $\Theta_S(V_\pi)$.

We now apply the standard estimate $\mathrm{tr}[XY] \leq \|X\|_1 \cdot \|Y\|$, and use that taking a partial transpose of both X and Y does not change the trace. Hence, if $\|\Theta_S(A)\| \leq 1$,

$$\begin{aligned} d^{-4}|\mathrm{tr}[AV_\sigma]| &= d^{-4}|\mathrm{tr}[\Theta_S(A)\Theta_S(V_\sigma)]| \\ &\leq d^{-4}\|\Theta_S(V_\sigma)\|_1\|\Theta_S(A)\| \leq d^{-l_S(\sigma)}. \end{aligned} \tag{6}$$

Returning to the statement of the Lemma: Let π not be adapted to $\mathcal{P}$. Then there is some set S of the partition, which is not mapped into itself by π. For this set $l_S(\pi) \geq 1$. On the other hand, since A is admissible for $\mathcal{P}$ the inequality (3) must hold for this S, hence $\|\Theta_S(A)\| \leq 1$. Hence by combining (4) with (6) we get $|a_\pi| = d^{-4}|\mathrm{tr}[AV_\sigma]| + \mathbf{O}(1/d) \leq d^{-l_S(\pi)} + \mathbf{O}(1/d) = \mathbf{O}(1/d)$.

Tailoring the states.—The idea of the construction is to choose ρ_1 and ρ_0 so that $\mathrm{tr}[\rho_1 V_\pi] = \mathrm{tr}[\rho_0 V_\pi]$, for all permutations π which are adapted to *any* of the targeted hiding partitions $\mathcal{P}$. Thus when we insert (2) into $\mathrm{tr}[(\rho_1 - \rho_0)A]$ for any A admissible for $\mathcal{P}$, the only contributing coefficients are $a_\pi = \mathbf{O}(1/d)$. Hence the whole expectation goes to zero.

On the other hand, we will make sure that $\mathrm{tr}[(\rho_1 - \rho_0)V_\pi] \neq 0$, for at least one permutation adapted to each of the targeted revealing partitions. From this we get an admissible analyzing operator with analyzing quality $\varepsilon_2 \neq 0$, and independent of d. Analysis may not be with probability one, but imperfect analysis can always be upgraded to certainty as described in the following section.

Verifying the examples.—In the following examples the hiding states are given in terms of the vector of expectations in (1). The hiding partitions in each example are the given partition, together with all its permutations and all its refinements.

Weakest hiding: The only permutation adapted to the finest partition $\mathcal{P} = (\{1\}, \{2\}, \{3\}, \{4\})$ is the identity. Hence *any* way of fixing the expectations of permutation operators gives a hiding pair of states. For example, we can take ρ_0 (ρ_1) as the normalized projection to the Bose (= symmetric) subspace [the Fermi (= antisymmetric) subspace] of $(\mathbb{C}^d)^{\otimes 4}$. Thus $\vec{\rho}_0 = (1, 1, 1, 1)$ and $\vec{\rho}_1 = (-1, 1, 1, -1)$. Obviously, if just two partners, e.g., 1 and 2, can exchange quantum information they can find out which alternative, 0 or 1, was chosen by just looking at the restriction of the state to their pair of subsystems, and measuring "symmetry" $A = (\mathbb{1} + V_{12})/2$.

Hiding against single pairs: For all pair partitions $\mathcal{P} = (\{1, 2\}, \{3\}, \{4\})$ the states $\vec{\rho}_0 = \frac{1}{3}(-1, -1, 0, 1)$ and $\vec{\rho}_1 = \frac{1}{3}(-1, 3, 0, -1)$ are hiding. Analysis for "single pairs" and "triplets" (see below) is imperfect.

Hiding against two pairs: For all partitions like $\mathcal{P} = (\{1, 2\}, \{3, 4\})$, the states $\vec{\rho}_0 = (0, 1, 1, 0)$ and $\vec{\rho}_1 = (0, 1, -\frac{1}{2}, 0)$ are hiding. However, a partition $(\{1, 2, 3\}, \{4\})$ can use $A = \frac{1}{3}[\mathbb{1} + V_{(123)} + V_{(321)}]$, to distinguish these with certainty.

Hiding against triplets: Conversely, the states $\vec{\rho}_0 = \frac{1}{3}(3, 1, 0, 3)$ and $\vec{\rho}_1 = \frac{1}{3}(1, -1, 0, -1)$ are hiding for any partition like $\mathcal{P} = (\{1, 2, 3\}, \{4\})$, but can be analyzed (imperfectly) by two pairs.

Strongest hiding: Finally, the states $\vec{\rho}_0 = \frac{1}{4}(0, 0, 1, 2)$ and $\vec{\rho}_1 = \frac{1}{4}(0, 0, 1, -2)$ are hiding unless quantum communication between all parties is established, in which case they can be distinguished perfectly.

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As these examples show, our construction so far does not guarantee perfect distinction ($\varepsilon_2 = 0$) for the partitions meant to be revealing. However, there is a single device to boost the detection quality, namely, to distribute several, say, K copies of the N-particle system, all prepared in the same state. Then for the hiding partitions we still get $\varepsilon_1 = \mathbf{O}(1/d)$. On the other hand, for the revealing partitions we can use detection operators A which are linear combinations of permutations. Then the detection probabilities $\mathrm{tr}[\rho_1 A]$ and $\mathrm{tr}[\rho_0 A]$ are independent of d, and if they are at all different, measuring A on all K copies distinguishes ρ_1 and ρ_0 with any desired degree of certainty.

This shows that for getting good discrimination $\varepsilon_2 \to 1$ we do not really need orthogonal states. What counts is that ρ_0 and ρ_1 are different along appropriate directions. Thus they can even be chosen to be close to the maximally mixed state and, in particular, separable. Since this was conjectured in [4,5] we include an explicit example, namely, the bipartite ($N = 2$) case of our construction. At the same time this illustrates nicely the interplay between the parameters d and K.

We use a simplified (but slightly weaker) bound to establish hiding: Since all admissible analyzing operators satisfy $\mathbf{0} \leq \Theta_{\{2\}}(A) \leq \mathbb{1}$, we get $|\mathrm{tr}[(\rho_1 - \rho_0)A]| = |\mathrm{tr}[\Theta_{\{2\}}(\rho_1 - \rho_0)\Theta_{\{2\}}(A)]| \leq \|\Theta_{\{2\}}(\rho_1 - \rho_0)\|_1$.

Our single copy scheme is based on bipartite Werner states. With $P_\pm$ the antisymmetric/symmetric projectors on $\mathbb{C}^d \otimes \mathbb{C}^d$ and $\rho_\pm = P_\pm/\mathrm{tr}[P_\pm]$ our hiding states are

$$\hat{\rho}_0 = \left(\frac{\rho_+ + \rho_-}{2}\right)^{\otimes K}, \qquad \hat{\rho}_1 = \rho_+^{\otimes K}, \tag{7}$$

which are clearly separable [6]. From this one can readily compute the partial transposes $\Theta_{\{2\}}(\rho_i)^{\otimes K}$ and their trace norm difference, as well as the expectations of the analyzing operator $A = P_+^{\otimes K}$, to get

$$\varepsilon_1 = 1 - (1 - 1/d)^K, \quad \text{and} \quad \varepsilon_2 = 2^{-K}. \tag{8}$$

Thus we can first choose K large to make ε_2 small and, subsequently d large, to get $\varepsilon_1 = K/d + \mathbf{O}(d^{-2})$ small.

This separable scheme is remarkably robust even if the analyzing partners share some entanglement: If they share a maximally entangled pair of a D-dimensional system with fixed D, we get the same asymptotic behavior in the limit $d \to \infty$, just with worse constants. Only if we choose D to grow on the same scale as d, i.e., on the same scale which would make teleportation possible, do we find that hiding becomes impossible.

Funding by the European Union Project EQUIP (Contract No. IST-1999-11053) and financial support from the DFG (Bonn) are gratefully acknowledged.

*Electronic address: t.eggeling@tu-bs.de
†Electronic address: r.werner@tu-bs.de

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