Conditions for the Local Manipulation of Gaussian States

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We present a general necessary and sufficient criterion for the possibility of a state transformation from one mixed Gaussian state to another of a bipartite continuous-variable system with two modes. The class of operations that will be considered is the set of local Gaussian completely positive trace-preserving maps.

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Imagine a physical device that is able to manipulate locally the state of a composite quantum system by actions on its parts. Which state transformations could this device implement in principle, abstracting from experimental imperfections? This question is particularly important in the field of quantum information theory [1], which concerns itself with the problem of whether a certain resource, e.g., an entangled quantum system in a known state, could be used to accomplish an envisioned task. To be more specific, one asks for mathematical conditions that have to be met in order for a state transformation under natural constraints to be possible.

Such a natural constraint is that the device only can implement local quantum operations supplemented by classical communication (LOCC), as many applications in quantum information science involve spatially separated parties sharing entangled states. So far, when investigating transformation criteria under LOCC all efforts have been devoted to the case where the involved quantum systems possess finite-dimensional Hilbert spaces as, e.g., qubit systems. The widely acknowledged result of Ref. [2] relates the problem of the deterministic transformation between pure states by means of LOCC to the mathematical theory of majorization. Based on this insight a series of further results has been found [3,4]. While the constraint to general LOCC is natural for low-dimensional systems, the situation is quite different for systems with an infinite-dimensional Hilbert space, such as the modes of an electromagnetic field. The experimental operations that are typically available are those that involve beam splitters, phase shifters, and squeezers together with the ability to prepare ancilla systems in a standard state such as the vacuum. The class of states that can be generated by these operations, and which is therefore particularly relevant from an experimental point of view, is the set of Gaussian states [6–9]. Several properties of entangled Gaussian states are already known. In particular, the problems of distillability and separability of Gaussian states have been investigated in great detail and can actually be considered solved [6,7]. However, the general question of the local interconvertability between entangled Gaussian states has not been addressed before.

This Letter presents a first step towards finding tools for deciding whether a desired transformation of Gaussian states can be accomplished without the need of going through all physical protocols, which can be an extremely tedious task. We will present a general necessary and sufficient criterion for the possibility of state transformations of a two-mode continuous-variable system. The class of allowed operations is the set of local Gaussian (non-measuring) completely positive maps [10], that is, those local operations that can be realized by means of local joint symplectic transformations on both the system and arbitrary appended ancilla systems that have been prepared in Gaussian states. This set will be abbreviated as LOG, and the statement that a transformation from a state ρ, pure or mixed, to a state ρ' is possible will be written as

$$\rho \rightarrow \rho' \text{ under LOG.}$$

In quantum optical systems this class of operations can be realized with present technology as a combination of applications of beam splitters, phase shifts, and squeezers together with the possibility to append additional field modes locally.

The physical system under consideration is a bipartite quantum system with 1 canonical degree of freedom each, such as two modes of an electromagnetic field. As in Ref. [7] such a system will be called a 1×1 system, consisting of parts 1 and 2. In order to exploit the elegant

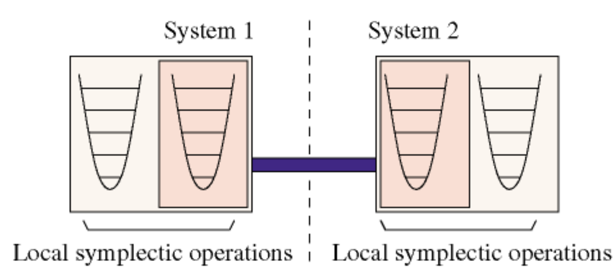

FIG. 1 (color online). Any local Gaussian completely positive map can be conceived as a composition of a local joint symplectic transformation on both the system and additional oscillators which have been prepared in a Gaussian state and a partial trace operation with respect to the additional oscillators.

formalism that is available to describe Gaussian quantum states [6,8] it is convenient to group the Hermitian operators corresponding to position and momentum in a vector, $O = (X_1, P_1, X_2, P_2)$. The canonical commutation relations (CCR) can then be subsumed into the skew symmetric block diagonal 4×4 matrix Σ according to $[O_n, O_m] = i\Sigma_{nm},\ n, m = 1, \ldots, 4$. For a given state ρ, for which the second moments exist, let the real 4×4 matrix Γ be defined as

$$\Gamma_{nm} = 2\mathrm{tr}[\rho(O_n - \langle O_n\rangle_\rho)(O_m - \langle O_m\rangle_\rho)] - i\Sigma_{nm}$$

where $\langle O_n\rangle_\rho = \mathrm{tr}[\rho O_n]$. The matrix Γ will be referred to as the covariance matrix. Not all symmetric 4×4 matrices are legitimate covariance matrices: the restriction that ρ is a state manifests itself as the condition $\Gamma - i\Sigma \geq 0$ for the covariance matrix, which is in fact a formulation of the uncertainty relations. For Gaussian states [12] the covariance matrix together with the mean values of the position and momentum operators is sufficient to fully specify the state. The first moments, however, are of no relevance for the issue of this paper, because they can always be made to vanish by an appropriate local translation in phase space.

We will now turn to the possible state transformations. Of particular interest are the linear transformations from one set of canonical coordinates to another set which leave the CCR invariant. In a system with 2 canonical degrees of freedom they form the group of real symplectic transformations $Sp(4, \mathbb{R})$ [8]. The group $Sp(4, \mathbb{R})$ consists of the real 4×4 matrices S obeying $S^T\Sigma S = \Sigma$; the group $Sp(2N, \mathbb{R})$ can be defined in an analogous manner for N canonical degrees of freedom. Under a symplectic transformation a covariance matrix is transformed according to $\Gamma \longmapsto S^T\Gamma S$. On the level of states it is accompanied by a unitary operation $\rho \longmapsto U(S)\rho U(S)^\dagger$, then called symplectic operation. A local symplectic transformation is a matrix S of the form $S = S_1 \oplus S_2$, where $S_1, S_2 \in Sp(2, \mathbb{R})$. The most general LOG can now be conceived as a composition of a joint symplectic transformation $S = S_1 \oplus S_2$ with $S_1, S_2 \in Sp(2N + 2, \mathbb{R})$ on the original systems 1 and 2 and on two additional systems with N canonical degrees of freedom each of which has been locally prepared in a Gaussian state and a partial trace operation with respect to the additional systems (see Fig. 1).

Any covariance matrix Γ of a bipartite 1×1 system can be written in block form as

$$\Gamma = \begin{pmatrix} A_1 & B \\ B^T & A_2 \end{pmatrix}, \tag{1}$$

where A_1, A_2, and B are real 2×2 matrices [13]. One can uniquely characterize the orbit $O(\Gamma)$ of Γ with respect to local symplectic transformations by a vector $(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}$, the entries of which are given by $\xi_1 := |A_1|^{1/2} \geq 1$, $\xi_2 := |A_2|^{1/2} \geq 1$, where $|\cdot|$ denotes the determinant. ξ_3 and ξ_4 are the solutions of $\xi_3\xi_4 = |B|$, $\xi_3^2 + \xi_4^2 = (|B|^2 - |\Gamma| + |A_1||A_2|)/(|A_1||A_2|)^{1/2}$, such that $\xi_3 \geq |\xi_4|$. It has been shown in Ref. [6] that Γ can always be transformed into a covariance matrix $S^T\Gamma S$ which is of "normal form" by using an appropriate local symplectic transformation S: this means that $S^T\Gamma S$ is of the form of Eq. (1), but with $B = \mathrm{diag}(\xi_3, \xi_4)$ and $A_i = \mathrm{diag}(\xi_i, \xi_i)$, $i = 1, 2$.

Whether a transformation of a state ρ to a state ρ' with respective covariance matrices Γ and Γ' is possible or not, will turn out to be largely determined by two functions $f_1^{\Gamma\to\Gamma'}$, $f_2^{\Gamma\to\Gamma'}$ that will be called minimal functions for reasons that will become clear later. Let $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$,

$$g(a, b, c, d) := (a^2 - 1) + (b^2 - 1)c^2d^2 + 2cd - ab(c^2 + d^2).$$

For a pair (Γ, Γ') of covariance matrices with associated vectors $(\xi_1, \xi_2, \xi_3, \xi_4)$ and $(\xi_1', \xi_2', \xi_3', \xi_4')$ with $\xi_3, \xi_4 > 0$ define the two functions $f_1^{\Gamma\to\Gamma'}, f_2^{\Gamma\to\Gamma'} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ as

$$f_1^{\Gamma\to\Gamma'}(x, y) := g(\xi_1', \xi_1, x/\xi_3, y/\xi_4),$$
$$f_2^{\Gamma\to\Gamma'}(x, y) := g(\xi_2', \xi_2, \xi_3'/x, \xi_4'/y).$$

The first statement concerns LOG in system 1 only. In this case the criterion amounts to simple inequalities that have to be satisfied. The second gives the full result for general LOG.

Proposition 1. *Let ρ and ρ'' be Gaussian states of a 1×1 system with covariance matrices Γ and Γ'' and associated vectors $(\xi_1, \xi_2, \xi_3, \xi_4)$ and $(\xi_1'', \xi_2, \xi_3'', \xi_4'')$ with $\xi_4, \xi_4'' > 0$. Then $\rho \to \rho''$ under LOG in system 1, if and only if*

$$1.\ |\xi_3\xi_4|/\xi_1 \geq |\xi_3''\xi_4''|/\xi_1'', \quad 2.\ f_1^{\Gamma\to\Gamma''}(\xi_3'', \xi_4'') \geq 0.$$

Proposition 2. *Let ρ and ρ' be Gaussian states of a 1×1 system with covariance matrices Γ and Γ' and associated vectors $(\xi_1, \xi_2, \xi_3, \xi_4)$ and $(\xi_1', \xi_2', \xi_3', \xi_4')$ with $\xi_4, \xi_4' > 0$. Then $\rho \to \rho'$ under LOG, if and only if one of the points*

$$(x, y) \in (f_1^{\Gamma\to\Gamma'})^{-1}(0) \cap (f_2^{\Gamma\to\Gamma'})^{-1}(0)$$

satisfies $|\xi_3\xi_4|\xi_1'/\xi_1 \geq |xy| \geq |\xi_3'\xi_4'|\xi_2/\xi_2'$.

Proof of Proposition 1. We begin with investigating what conditions have to be met when a LOG is implemented in system 1 and a symplectic operation in system 2. The starting point is a general representation theorem concerning Gaussian completely positive maps [11]: Any Gaussian completely positive map is reflected on the level of the covariance matrix as a map

$$\Gamma \longmapsto M^T\Gamma M + G, \tag{2}$$

where M and G are real 4×4 matrices, and G is moreover symmetric. The condition

$$G + i\Sigma - iM^T\Sigma M \geq 0 \tag{3}$$

on the matrices M and G incorporates the complete positivity of the map. The state transformation mapping Γ on Γ'' can be decomposed into three steps: first, an appropriate matrix $S = S_1 \oplus S_2$, $S_1, S_2 \in Sp(2, \mathbb{R})$, is applied on the initial covariance matrix Γ, such that $S^T \Gamma S$ is of normal form. Then a LOG restricted to system 1 and a symplectic operation in system 2 is implemented, mapping $S^T\Gamma S$ onto another matrix in normal form. Finally, $T = T_1 \oplus T_2$, $T_1, T_2 \in Sp(2, \mathbb{R})$, is used in order to transform the resulting matrix into Γ''. The second step can be represented in the form of Eq. (2) with real matrices M and G. Clearly, the composition of the three steps, $\Gamma \longmapsto (T^T M^T S^T)\Gamma(SMT) + T^T G T$ amounts again to a LOG. Therefore, we can without loss of generality assume that both Γ and Γ'' are already in normal form with associated vectors $(\xi_1, \xi_2, \xi_3, \xi_4)$ and $(\xi_1'', \xi_2'', \xi_3'', \xi_4'')$. The task is then to find appropriate real matrices M and G as above such that $\Gamma'' = M^T \Gamma M + G$, representing a LOG restricted to system 1 and a symplectic operation in system 2. Hence, it is required that M and G are of the form $M = M_1 \oplus M_2$, $G = G_1 \oplus 0$, where G_1 is symmetric and $M_2 \in Sp(2, \mathbb{R})$, i.e., M_2 satisfies $M_2^T \Sigma M_2 = \Sigma$. Because of the normal form of Γ and Γ'' we have that $M_2^T \mathrm{diag}(\xi_2, \xi_2) M_2 = \mathrm{diag}(\xi_2, \xi_2)$, and it follows that $M_2 \in SO(2)$. Let us set $M_{33} = M_{44} = \cos(\theta/2)$, $M_{34} = -\sin(\theta/2)$, and $M_{43} = \sin(\theta/2)$ with $\theta \in (-2\pi, 2\pi]$. The requirement that $\Gamma'' = M^T \Gamma M + G$ implies then a certain set of equations that has to be satisfied, connecting the entries of M_1 and M_2. An elementary calculation yields finally $M_{11} = (\xi_3''/\xi_3)\cos(\theta/2)$, $M_{22} = (\xi_4''/\xi_4)\cos(\theta/2)$, $M_{12} = -(\xi_4''/\xi_3)\sin(\theta/2)$, $M_{21} = (\xi_3''/\xi_4)\sin(\theta/2)$. Not all such matrices $M = M_1 \oplus M_2$ and $G = \Gamma'' - M^T \Gamma M$ define a completely positive map, however. Because of the block diagonal form of M and G, the inequality (3) reflecting the complete positivity can be written as

$$H_1 := G_1 + i\Sigma(1 - |M_1|) \geq 0.$$

As H_1 is a Hermitian 2×2 matrix, $H_1 \geq 0$ is in turn equivalent to $|H_1| \geq 0$, $\mathrm{tr}[H_1] \geq 0$. The determinant $|M_1| = (\xi_3''\xi_4'')/(\xi_3\xi_4)$ is independent of θ. The determinant and trace of H_1 can be evaluated to $\mathrm{tr}[H_1] = 2\xi_1'' - \xi_1 M_1{}^2$ and $|H_1| = (\xi_1'')^2 - \xi_1\xi_1'' M_1{}^2 + \xi_1^2 |M_1|^2 - (1 - |M_1|)^2$, where M_1 denotes the Hilbert-Schmidt norm of M_1. Hence, it is always optimal to choose θ in such a way that

$$M_1{}^2 = \left[\frac{(\xi_3'')^2}{\xi_3^2} + \frac{(\xi_4'')^2}{\xi_4^2}\right]\cos^2\frac{\theta}{2} + \left[\frac{(\xi_4'')^2}{\xi_3^2} + \frac{(\xi_3'')^2}{\xi_4^2}\right]\sin^2\frac{\theta}{2}$$

is minimal. But since $\xi_3^2 \geq \xi_4^2$ and $(\xi_3'')^2 \geq (\xi_4'')^2$, it is true that always $(\xi_3''/\xi_3)^2 + (\xi_4''/\xi_4)^2 \leq (\xi_4''/\xi_3)^2 + (\xi_3''/\xi_4)^2$, and therefore, $\theta = 0$ is the optimal choice. To simplify the structure of the requirements one can proceed as follows: The inequality $|H_1| \geq 0$ implies, in particular, that $\xi_1''/\xi_1 \geq |\xi_3''\xi_4''|/|\xi_3\xi_4|$. Whenever this inequality is satisfied, $|H_1| \geq 0$ yields a stronger upper bound for $M_1{}^2$ as $\mathrm{tr}[H_1] \geq 0$ does, as then

$$[(\xi_1'')^2 + \xi_1^2|M_1| - (1 - |M_1|)^2]/(\xi_1''\xi_1) \leq 2\xi_1''/\xi_1. \quad (4)$$

Altogether, this implies that equivalently to requiring the validity of both $|H_1| \geq 0$ and $\mathrm{tr}[H_1] \geq 0$ one may require that both $\xi_1''/\xi_1 \geq |\xi_3''\xi_4''|/|\xi_3\xi_4|$ and $|H_1| \geq 0$ hold. Therefore, we finally arrive at the statement that $\rho \to \rho''$ under a LOG in system 1 and a symplectic operation in system 2 if and only if both $|\xi_3''\xi_4''|/\xi_1' \leq |\xi_3\xi_4|/\xi_1$ and $f_1^{\Gamma\to\Gamma''}(\xi_3'', \xi_4'') \geq 0$ are satisfied. This criterion depends only on the invariants with respect to local symplectic operations, and hence, we arrive at Proposition 1. □

It is worth noting what physical situation is reflected by equality $f_1^{\Gamma\to\Gamma''}(\xi_3'', \xi_4'') = 0$. It can easily be shown that equality holds if and only if M and G satisfy $G = K^T G^{-1} K$, where $K := M^T \Sigma M - \Sigma$. Solutions of this type are the minimal solutions in the sense of Ref. [9]: For a given initial covariance matrix Γ and a given matrix M the equation $G = K^T G^{-1} K$ specifies those symmetric matrices G that add minimal noise. Such LOG will consequently be called minimal. Hence, a LOG in system 1 from ρ to ρ'' is minimal if and only if $f_1^{\Gamma\to\Gamma''}(\xi_3'', \xi_4'') = 0$ holds (see Fig. 2). Therefore, one may interpret the conditions of Proposition 1 in physical terms as follows: the first condition requires that the "stretching" $|M|$ is sufficiently small, the second makes sure that enough noise is introduced in the course of the transformation.

Proof of Proposition 2. A general LOG can again be decomposed into several steps. As before, without loss of generality one may assume that the initial and the final covariance matrices Γ and Γ' are of normal form with associated vectors $(\xi_1, \xi_2, \xi_3, \xi_4)$ and $(\xi_1', \xi_2', \xi_3', \xi_4')$, respectively. In two intermediate steps one transforms $\Gamma \mapsto \Gamma''$ and $\Gamma'' \mapsto \Gamma'$ by means of LOG restricted to one system and appropriate symplectic operations in the other system. The vector associated with Γ'' will be denoted as (ξ_1', ξ_2, x, y). One can proceed as before, and after applying analogous steps one finally arrives at the criterion that $\rho \to \rho'$ under LOG if and only if there exists an $(x, y) \in \mathbb{R}^+ \times \mathbb{R}$ such that the inequalities

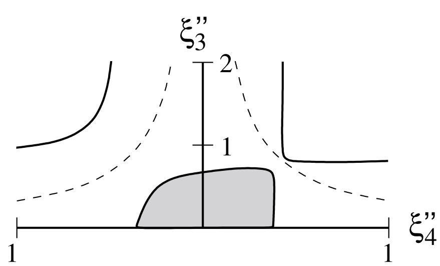

FIG. 2. Given is a state ρ with a covariance matrix Γ with associated vector $(\xi_1, \xi_2, \xi_3, \xi_4) = (3, 5, 1, 1/2)$. The shaded area depicts what values of ξ_3'' and ξ_4'' are accessible under a LOG in system 1, under the assumption that the final covariance matrix Γ'' is associated with a vector $(\xi_1'', \xi_2, \xi_3'', \xi_4'')$ with $\xi_1'' = 2$. The thick line corresponds to those points (ξ_3'', ξ_4'') with $f_1^{\Gamma\to\Gamma''}(\xi_3'', \xi_4'') = 0$ for which the transformation is a minimal LOG; the dashed line represents the points satisfying $|\xi_3''\xi_4''| = |\xi_3\xi_4|\xi_1''/\xi_1$.

$$w_1 \geq |xy| \geq w_2, \qquad f_1^{\Gamma\to\Gamma'}(x,y) \geq 0, \qquad f_2^{\Gamma\to\Gamma'}(x,y) \geq 0 \tag{5}$$

are simultaneously satisfied, where $w_1 := |\xi_3\xi_4|\xi_1'/\xi_1$ and $w_2 := |\xi_3'\xi_4'|\xi_2/\xi_2'$. This is already a criterion in its own, but it still requires a search in a two-dimensional set. The key observation in a simplification is that the intersection of the interior of the set $(f_i^{\Gamma\to\Gamma'})^{-1}(\mathbb{R}^+)$ and the set $N_i := \{(x,y) \in \mathbb{R}^+ \times \mathbb{R} : |xy| = w_i\}$ is empty for both $i = 1, 2$ (see Fig. 2). The minimal value of $(x/\xi_3)^2 + (y/\xi_4)^2$ for $(x,y) \in N_1$ is given by $2\xi_1'/\xi_1$. Hence, it follows from Eq. (4) that $f_1(x,y) < 0$ for all $(x,y) \in N_1$, if $\xi_1 \neq \xi_1'$, and $f_1(x,y) \leq 0$ for all $(x,y) \in N_1$, if $\xi_1 = \xi_1'$. Similarly, $f_2(x,y) \leq 0$ for all $(x,y) \in N_2$. Moreover, $f_1^{\Gamma\to\Gamma'}$ is continuous on $\mathbb{R} \times \mathbb{R}^+$, and for $f_2^{\Gamma\to\Gamma'}$ there exists a continuous extension on $\mathbb{R}^+ \times \mathbb{R}$. The problem is therefore reduced to the subsequent search for intersection points: there exists an $(x,y) \in \mathbb{R}^+ \times \mathbb{R}$ satisfying (5) if and only if there exists a point $(x,y) \in (f_1^{\Gamma\to\Gamma'})^{-1}(0) \cap (f_2^{\Gamma\to\Gamma'})^{-1}(0)$ such that $w_1 \geq |xy| \geq w_2$. This is Proposition 2. In particular, this means that if the transformation $\rho \longrightarrow \rho'$ is possible, it can always be realized as a composition of two minimal LOG in systems 1 and 2, respectively, [14]. □

So far, the simple case has been omitted that the initial state ρ has a covariance matrix Γ with associated vector $(\xi_1, \xi_2, \xi_3, \xi_4)$, where $\xi_4 = 0$. It turns out that one can proceed as before. In the notation of Proposition 1 (but with $\xi_4 = 0$), one arrives at the statement that $\rho \longrightarrow \rho''$ under LOG restricted to system 1, if and only if both $\xi_4'' = 0$ and $(\xi_3''/\xi_3)^2 \leq [(\xi_1'')^2 - 1]/(\xi_1\xi_1'')$. Consequently, in the notation of Proposition 2, $\rho \longrightarrow \rho'$ under LOG, if and only if $\xi_4'' = 0$ and $(\xi_3''/\xi_3)^2 \leq [(\xi_2')^2 - 1][(\xi_1')^2 - 1]/(\xi_1\xi_1'\xi_2\xi_2')$.

As a first application we can look for Gaussian states ρ and ρ' that are incommensurate, that is, pairs of states (ρ, ρ') for which neither $\rho \longrightarrow \rho'$ under LOG nor $\rho' \longrightarrow \rho$ under LOG holds. As can readily be verified using Proposition 2, an example of such a pair is given by states specified by covariance matrices associated with $(2, 2, 1, 1)$, $(2, 2, 1, -1/2)$, respectively. The relation that a state can be transformed into another state under LOG induces hence a partial order on the set of Gaussian states, but not a total order.

With this Letter we have posed and answered a basic question: Under the constraint of locality, we ask which pairs of Gaussian states allow for a transformation from one state to the other. The choice for the set of allowed operations—Gaussian completely positive maps [10]—has been motivated by pragmatic considerations: in quantum optical systems such operations can be implemented with present technology. Needless to say, there are many open questions that may be approached with similar methods. In particular, one may take into account selective measurements projecting on Gaussian states, such as in homodyne detection, together with classical communication. It is the hope that this Letter stimulates such further considerations.

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[1] R. F. Werner, in *Quantum Information*, Springer Tracts in Modern Physics Vol. 173 (Springer, Heidelberg, 2001).
[2] M. A. Nielsen, Phys. Rev. Lett. **83**, 436 (1999).
[3] G. Vidal, Phys. Rev. Lett. **83**, 1046 (1999); D. Jonathan and M. B. Plenio, *ibid.* **83**, 1455 (1999); L. Hardy, Phys. Rev. A **60**, 1912 (1999).
[4] The analysis of state transformations of mixed states is much more involved, and only isolated results are known [5].
[5] J. Eisert and M. Wilkens, Phys. Rev. Lett. **85**, 437 (2000); P. Hayden, B. M. Terhal, and A. Uhlmann, quant-ph/0011095.
[6] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000).
[7] L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. 84, 2722 (2000); R. F. Werner and M. M. Wolf, *ibid.* **86**, 3658 (2001); G. Giedke, B. Kraus, M. Lewenstein, and J. I. Cirac, *ibid.* **87**, 167904 (2001); N. J. Cerf, A. Ipe, and X. Rottenberg, *ibid.* **85**, 1754 (2000); S. Parker, S. Bose, and M. B. Plenio, Phys. Rev. A **61**, 032305 (2000); G. Giedke, L.-M. Duan, P. Zoller, and J. I. Cirac, Quant. Inf. Comp. **1**, 79 (2001).
[8] R. Simon, E. C. G. Sudarshan, and N. Mukunda, Phys. Rev. A **36**, 3868 (1987).
[9] G. Lindblad, J. Phys. A **33**, 5059 (2000).
[10] The term Gaussian completely positive map is used here as in the mathematical physics literature [9,11], as the generalization of convolutions of Gaussian measures in the commutative case. This set includes all possible completely positive trace-preserving operations that map Gaussian states on Gaussian states, under exclusion of those operations that include selective measurements projecting on the improper eigenstates of a canonical coordinate, together with further operations depending on the (continuous) measurement outcome.
[11] B. Demoen, P. Vanheuverzwijn, and A. Verbeure, Lett. Math. Phys. **2**, 161 (1977).
[12] A state ρ is called Gaussian, if the characteristic function ϕ, defined as a function of $\mathbf{z} = (x_1, p_1, x_2, p_2)$ as $\phi(\mathbf{z}) := \mathrm{tr}[D(\mathbf{z})\rho]$, $D(\mathbf{z}) := \exp(i\sum_{n,m=1}^4 z_n \Sigma_{nm} O_m)$, is a Gaussian function in phase space.
[13] The four numbers $|A_1|$, $|A_2|$, $|B|$, and $|\Gamma|$ are invariant under all $S \in Sp(4, \mathbb{R})$ of the form $S = S_1 \oplus S_2$.
[14] Practically, it is helpful to consider explicit parametrizations of $(f_1^{\Gamma\to\Gamma'})^{-1}(0)$ and $(f_2^{\Gamma\to\Gamma'})^{-1}(0)$: there exist sets C_i and two functions $x_i^{\pm} : C_i \longrightarrow \mathbb{R}^+$ each which satisfy $(f_i^{\Gamma\to\Gamma'})[x_i^{\pm}(y), y] = 0$ identically. One has then to look for the intersection points of these explicit parametrizations.