Replica Field Theory and Renormalization Group for the Ising Spin Glass in an External Magnetic Field

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We use the generic replica symmetric cubic field theory to study the transition of short-range Ising spin glasses in a magnetic field around the upper critical dimension. A novel fixed point is found from the application of the renormalization group. In the spin-glass limit, this fixed point governs the critical behavior of a class of systems characterized by a single cubic parameter. For this universality class, the spin-glass susceptibility diverges at criticality, whereas the longitudinal mode remains massive. The third mode, however, behaves unusually. The physical consequences of this unusual behavior are discussed, and a comparison with the conventional de Almeida–Thouless scenario is presented.

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The mean-field theory of the Ising spin glass [1] provided an astounding complexity of equilibrium properties, showing how disorder and frustration may lead to an unusual thermodynamics. More than two decades have elapsed since the ultrametric solution of the Sherrington-Kirkpatrick [2] model by Parisi was published in a series of papers (see [1] for the references); nevertheless, consensus has not been reached about the validity of the mean-field picture for finite-dimensional, short-range systems. The alternative scenario, the so-called droplet picture [3,4], claims that the complex phase space structure is an artifact of mean-field theory; the glassy state consists of two phases related by the global inversion symmetry of the spins.

The investigation of the spin-glass transition in an external magnetic field may resolve the debate: The glassy transition along the de Almeida–Thouless (AT) line [5] is a distinctive feature of mean-field theory, whereas spin-glass ordering is destroyed by any nonzero magnetic field in the droplet model. Although a lot of numerical work has been performed, no convincing evidence has emerged until now in favor of either theory. An AT line was found in the four-dimensional case in Refs. [6,7], whereas numerical results in three dimensions were interpreted, although less convincingly, to support mean-field-like behavior in [6,8,9]. On the other side, Ref. [10] interprets the simulation data of [8] as quite consistent with droplet theory, and an analysis of the ground states in [11] showed that the spin-glass phase of the three-dimensional model does not survive in any finite magnetic field. More recently, however, an extensive study of the energy landscape [12] suggests that a nonzero critical field may exist at zero temperature, separating the spin-glass and paramagnetic phases.

In this Letter, replica field theory, as an alternative to numerical calculations, is used to attack the problem by extending the renormalization group study of Ref. [13]. Our starting point is the Edwards-Anderson [14] model of N Ising spins on a d-dimensional hypercubic lattice, defined by the Hamiltonian

$$\mathcal{H} = -\sum_{\langle ij\rangle} J_{ij} s_i s_j - H\sum_i s_i, \tag{1}$$

where the first (second) summation is over all nearest neighbor pairs $\langle ij\rangle$ (all lattice sites i), respectively. J_{ij} are independent, Gaussian distributed random variables with mean zero and variance Δ^2, and a homogeneous magnetic field H has also been included. The application of the replica trick followed by a Hubbard-Stratonovich transformation produces a *replica symmetric* field theory, with the Lagrangian $\mathcal{L}_{\rm micr}$ taking over the role of $\mathcal{H}$. In this replica field theory, the fields depend on two replica indices with the restriction $\phi^{\alpha\beta} = \phi^{\beta\alpha}$ and $\phi^{\alpha\alpha} \equiv 0$; hence, we have $n(n-1)/2$ field components. The replica number n must go to zero to reproduce quenched averages; we will argue, however, that it is necessary to keep it finite until the very end of the calculations [15]. As is common in the theory of phase transitions, the microscopic Lagrangian $\mathcal{L}_{\rm micr}$ is replaced by an effective one $\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(3)}$, obtained by iterating the renormalization group until irrelevant operators can be neglected. As a result of the replica trick, a *generic replica symmetric* field theory follows, which can be best represented in terms of operators invariant under the permutation of the n replicas:

$$\mathcal{L}^{(2)} = \frac{1}{2}\sum_{\mathbf{p}}\left[\left(\frac{1}{2}p^2 + m_1\right)\sum_{\alpha\beta}\phi_{\mathbf{p}}^{\alpha\beta}\phi_{-\mathbf{p}}^{\alpha\beta} + m_2\sum_{\alpha\beta\gamma}\phi_{\mathbf{p}}^{\alpha\gamma}\phi_{-\mathbf{p}}^{\beta\gamma} + m_3\sum_{\alpha\beta\gamma\delta}\phi_{\mathbf{p}}^{\alpha\beta}\phi_{-\mathbf{p}}^{\gamma\delta}\right], \tag{2}$$

$$\mathcal{L}^{(3)} = -\frac{1}{6\sqrt{N}} \sum_{\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3}{}' \Bigg[w_1 \sum_{\alpha\beta\gamma} \phi^{\alpha\beta}_{\mathbf{p}_1} \phi^{\beta\gamma}_{\mathbf{p}_2} \phi^{\gamma\alpha}_{\mathbf{p}_3} + w_2 \sum_{\alpha\beta} \phi^{\alpha\beta}_{\mathbf{p}_1} \phi^{\alpha\beta}_{\mathbf{p}_2} \phi^{\alpha\beta}_{\mathbf{p}_3} + w_3 \sum_{\alpha\beta\gamma} \phi^{\alpha\beta}_{\mathbf{p}_1} \phi^{\alpha\beta}_{\mathbf{p}_2} \phi^{\alpha\gamma}_{\mathbf{p}_3} + w_4 \sum_{\alpha\beta\gamma\delta} \phi^{\alpha\beta}_{\mathbf{p}_1} \phi^{\alpha\beta}_{\mathbf{p}_2} \phi^{\gamma\delta}_{\mathbf{p}_3}$$
$$+ w_5 \sum_{\alpha\beta\gamma\delta} \phi^{\alpha\beta}_{\mathbf{p}_1} \phi^{\alpha\gamma}_{\mathbf{p}_2} \phi^{\beta\delta}_{\mathbf{p}_3} + w_6 \sum_{\alpha\beta\gamma\delta} \phi^{\alpha\beta}_{\mathbf{p}_1} \phi^{\alpha\gamma}_{\mathbf{p}_2} \phi^{\alpha\delta}_{\mathbf{p}_3} + w_7 \sum_{\alpha\beta\gamma\delta\mu} \phi^{\alpha\gamma}_{\mathbf{p}_1} \phi^{\beta\gamma}_{\mathbf{p}_2} \phi^{\delta\mu}_{\mathbf{p}_3}$$
$$+ w_8 \sum_{\alpha\beta\gamma\delta\mu\nu} \phi^{\alpha\beta}_{\mathbf{p}_1} \phi^{\gamma\delta}_{\mathbf{p}_2} \phi^{\mu\nu}_{\mathbf{p}_3} \Bigg]. \tag{3}$$

The above field-theoretical model is general enough to describe a large variety of possible transitions from the high temperature, replica symmetric phase, just below the upper critical dimension $d = 6$. The failure of Bray and Roberts [13] to detect a fixed point corresponding to the conventional AT transition gave definite support, from the analytic part, to the droplet theory. In this Letter, we present a novel fixed point characterized by a rather unusual property and propose it as the relevant one for the *generic* replica symmetric phase, i.e., that with a nonzero order parameter. We also put forward a possible physical scenario whose validity is a prerequisite for that fixed point to control the spin-glass transition in a magnetic field. Checking this, however, is out of the scope of this Letter, and we leave it for future work.

Even a leading order renormalization group calculation is blocked by the difficulties arising from the numerous and complicated replica summations and from the fact that $\mathcal{L}^{(2)}$ is not in a diagonalized form. In Ref. [17] we worked out a transformation to a new set of bare parameters (r_R, r_A, and r_L for the masses and $g_1, \ldots, g_8$ for the cubic couplings) rendering the one-loop calculation feasible. As an illustration, we computed in [17] the true masses Γ_R, Γ_A, and Γ_L to one-loop order. To obtain the renormalized cubic interaction, we calculated, via long but relatively straightforward algebra, the triangle graph. In order to be completely parallel to Ref. [13], we chose the same renormalization scheme of integrating out degrees of freedom in the infinitesimal momentum shell between $e^{-dl}\Lambda$ and Λ, where Λ is the ultraviolet cutoff. It is obvious that there is no sufficient space to present here the recursion relations in their total generality, as worked out in [19]. The structure of these renormalization-group (RG) equations is as follows:

$$dr_i/dl = \mathcal{R}_i(r_R, r_A, r_L; g_1, \ldots, g_8), \qquad i = R,\ A,\ \text{or}\ L;$$
$$dg_i/dl = \mathcal{G}_i(r_R, r_A, r_L; g_1, \ldots, g_8), \qquad i = 1, \ldots, 8. \tag{4}$$

The above set of equations must comprise two special cases, known from the literature for some time, providing us with a good check:

(i) The zero magnetic field case was studied in Ref. [20] up to $O(\epsilon^3)$, the Lagrangian corresponding to it ($r_R = r_A = r_L \equiv r$, $w_1 \equiv w$, and $w_i = 0$ for $i = 2, \ldots, 8$) proves to be an invariant subspace of the set of Eqs. (4), expressing the higher symmetry this system possesses. The fixed point attracts a critical line in $w - r$ space, which is totally massless ($\Gamma_R = \Gamma_A = \Gamma_L = 0$), and, following the ideas in [17], we can identify this theory as the relevant field-theoretical model for the spin-glass transition from the paramagnet to the generic replica symmetric phase with a nonzero order parameter. The fixed-point value $w^{*2} = -\frac{1}{n-2}\epsilon$, with $\epsilon \equiv 6 - d$, and the exponents η and ν are in complete agreement, up to first order in ϵ (see [19]), with the results of Green [20] for $n = 0$. As in mean-field theory, n is a rather innocent parameter around this fixed point.

(ii) The equations of Bray and Roberts [13] are reproduced by assuming tentatively a critical surface with $\Gamma_R = 0$, whereas Γ_A and Γ_L are finite. At some hypothetical fixed point, the bare anomalous and longitudinal masses are infinite, resulting in a pair of recursion relations for the repliconlike couplings g_1 and g_2:

$$dg_i/dl = \bar{\bar{\mathcal{G}}}_i(g_1, g_2), \qquad i = 1, 2. \tag{5}$$

Using the relations $g_1 = w_1$ and $g_2 = 2w_2$ [17], we arrive at the RG recursions whose physically relevant fixed point was searched for in vain by Bray and Roberts [13].

A new theory, invariant under renormalization to $O(\epsilon)$ order, emerges if the condition of degeneracy between the longitudinal (L) and anomalous (A) modes is removed. The physical picture behind this may be the following: The masses characterizing a critical manifold are observable through correlation functions. For the original lattice system of Eq. (1), three distinct spin-glass correlation functions can be defined (see, e.g., in [3]), their zero momentum limits will be denoted by G_1, G_2, and G_3. The transition in a field can be characterized by the divergence of the spin-glass susceptibility, $\chi_{\mathrm{SG}} = G_1 - 2G_2 + G_3 = \Gamma_R^{-1}$, manifesting itself in the criticality of the replicon (R) mode, while the longitudinal mode, $G_1 - 4G_2 + 3G_3 = \Gamma_L^{-1}$, remains massive. As for the anomalous one, we have the exact relation for n small but finite:

$$\Gamma_A = \frac{\Gamma_L}{1 - n\Gamma_L(G_2 - \frac{3}{2}G_3)}. \tag{6}$$

In mean-field theory, the combination $G_2 - \frac{3}{2}G_3$ is also analytical along the AT line, leading to the following ratios for the *leading* singularities: $G_1 : G_2 : G_3 = 1 : 1/2 : 1/3$. Dimension-dependent subleading singularities of the form $\sim t^{-\mu}$ occur in finite dimensions, t being the reduced temperature, and an analysis of the perturbation expansion of the propagators shows that for the next-to-leading term,

μ is equal to $4 - d/2$, $d > 6$. As a result, a jump of the anomalous mass may develop below $d = 8$ [21], provided the longitudinal mode remains massive. From Eq. (6) it follows:

$$\Gamma_A \sim \begin{cases} \Gamma_L & \text{if } n \to 0 \text{ first,} \\ \frac{1}{n} t^{\mu} & \text{if } t \to 0 \text{ first.} \end{cases} \tag{7}$$

From the assumptions Γ_L finite and $\mu > 0$, and using (6), a jump in the anomalous mass necessarily follows in the limit $n \to 0$ even below $d = 6$. Whether or not these assumptions are correct can be tested by the RG Eqs. (4). To detect this behavior, we now search for a nontrivial fixed point with $\Gamma_R = \Gamma_A = 0$ and $\Gamma_L = \infty$. It is obvious from Eq. (7) that n is a crucial parameter now, and it must be finite when computing the RG flows, setting it to 0 only at the very end of the calculation [23]. The longitudinal bare mass is kept at its infinite fixed-point value, longitudinal-like couplings (g_4, g_7, and g_8; see Eq. (48) of Ref. [17]) decouple from the rest of Eq. (4). A massless renormalization scheme [massless with respect to the replicon (R) and anomalous (A) masses] can be deduced from the remaining part of (4):

$$dg_i/dl = \bar{\mathcal{G}}_i(g_1, g_2, g_3, g_5, g_6), \qquad i = 1, 2, 3, 5, 6. \tag{8}$$

In our one-loop calculation, $\bar{\mathcal{G}}_i$ is a cubic polynomial of its variables with coefficients which are rational functions of n. It turns out that $\bar{\mathcal{G}}_3$ and $\bar{\mathcal{G}}_6$ are always zero whenever $g_3 = g_6 = 0$ [19], indicating that the three-dimensional manifold so defined is an invariant subspace of the RG Eqs. (8). Instead of presenting long and cumbersome formulas for generic n, we display the recursion relations for g_1, g_2, and g_5 only in the $n \to 0$ limit:

$$\frac{dg_1}{dl} = \frac{1}{2}(\epsilon - 3\eta_R)g_1 + 14g_1^3 - 18g_1^2 g_2 + \frac{9}{2}g_1 g_2^2 + \frac{1}{8}g_2^3 - 8g_5^3, \tag{9a}$$

$$\frac{dg_2}{dl} = \frac{1}{2}(\epsilon - 3\eta_R)g_2 + 24g_1^2 g_2 - 30g_1 g_2^2 + \frac{17}{2}g_2^3, \tag{9b}$$

$$\frac{dg_5}{dl} = \frac{1}{2}(\epsilon - \eta_R - 2\eta_A)g_5 - 8g_1 g_5^2 + 8g_2 g_5^2 + 8g_5^3; \tag{9c}$$

$$\eta_R \equiv \frac{1}{3}\left(4g_1^2 - 8g_1 g_2 + \frac{11}{4}g_2^2 + 4g_5^2\right). \tag{9d}$$

(η_A above is proportional to n; i.e., it is zero here. It is shown only to display the generic structure of the RG equations.)

We found a novel nontrivial fixed point from Eqs. (9a)–(9c):

$$g_1^* = \sqrt{\epsilon}/2, \qquad g_2^* = \sqrt{\epsilon}, \qquad g_5^* = -\sqrt{\epsilon}/4. \tag{10}$$

The existence of this fixed point is due to the term g_5^3 and g_5^2 in Eqs. (9a) and (9d), respectively; omitting them, we just get back the Bray-Roberts Eqs. (5). It is remarkable that, in a sense, we have found in Eqs. (9) a generalization of the RG theory put forward in Ref. [13]. We must notice, however, that beside the replicon mode, the anomalous one is also critical on the manifold attracted by the fixed point (10).

The most striking feature of Eqs. (9a)–(9c) is that they coincide for $g_1 = \bar{w}$, $g_2 = 2\bar{w}$, and $g_5 = -\bar{w}/2$, providing the single parameter RG equation for $\bar{w}$:

$$\frac{d\bar{w}}{dl} = \frac{\epsilon}{2}\bar{w} - 2\bar{w}^3.$$

Translating this to the language of the w couplings in Eq. (3), $w_1 = w_2 = w_6 = \bar{w}$, and all the other w's are zero. The system with the single cubic operator

$$\bar{w}\left[\sum_{\alpha\beta\gamma} \phi_{\mathbf{p}_1}^{\alpha\beta}\phi_{\mathbf{p}_2}^{\beta\gamma}\phi_{\mathbf{p}_3}^{\gamma\alpha} + \sum_{\alpha\beta} \phi_{\mathbf{p}_1}^{\alpha\beta}\phi_{\mathbf{p}_2}^{\alpha\beta}\phi_{\mathbf{p}_3}^{\alpha\beta} + \sum_{\alpha\beta\gamma\delta} \phi_{\mathbf{p}_1}^{\alpha\beta}\phi_{\mathbf{p}_2}^{\alpha\gamma}\phi_{\mathbf{p}_3}^{\alpha\delta}\right] \tag{11}$$

rescales under RG, evolving into the fixed point $\bar{w}^* = \sqrt{\epsilon}/2$, where the corresponding eigenvalue is $\lambda_{\bar{w}} = -\epsilon$. This result can be compared with the zero-field case, where the similar rescaling property of the system expressed a higher symmetry than the permutation invariance of the n replicas. There is one important distinction we must notice, however: the rescaling behavior of the zero-field Lagrangian under iteration is independent of n, whereas it develops only in the spin-glass limit $n \to 0$ for the system with the cubic coupling in (11).

Including the masses into the RG scheme, it can be easily checked that the condition $r_R = r_A \equiv \bar{r}$ is preserved under iteration. Using results from Ref. [17], namely, Eqs. (22)–(24) and (28)–(30), $m_1 = \bar{r}/2$ and $m_2 = 0$ follow then, while m_3 is infinite, inducing the freezing-out of the longitudinal component of $\phi^{\alpha\beta}$. The quadratic operator in the brackets of Eq. (2) reduces to the simple repliconlike invariant

$$(p^2 + \bar{r})\sum_{\alpha<\beta} \phi_{\mathbf{p}}^{\alpha\beta}\phi_{-\mathbf{p}}^{\alpha\beta}, \tag{12}$$

although $\phi^{\alpha\beta}$ has now an anomalous component too.

We can deduce critical indices belonging to this new fixed point, and we display them here for completeness:

$$\eta_R = O(\epsilon^2), \qquad \lambda_R = \nu_R^{-1} = 2 - \frac{\epsilon}{2} + O(\epsilon^2);$$
$$\eta_A = O(\epsilon^2), \qquad \lambda_A = \nu_A^{-1} = 2 + O(\epsilon^2).$$

To connect these to usual exponents like that of the spin-glass susceptibility, $\chi_{\mathrm{SG}} \sim t^{-\gamma}$, is not trivial now due to the coexistence of two critical masses, and needs further study.

We propose the simple model of Eqs. (11) and (12) as a candidate for studying the replica symmetry breaking (RSB) transition from the replica symmetric phase with a nonzero order parameter; the spin-glass transition in an external field belongs to this class. Nevertheless, it is difficult to find evidence for this. The check by testing the crossover, *from* a $\mathcal{L}_{\rm micr}$ in the vicinity of the zero-field critical point *to* the new fixed point, is blocked by the large distance in this huge parameter space and by the evolution of the longitudinal mass from the near-zero value to infinity. It is also obvious that the irrelevant operators present in $\mathcal{L}_{\rm micr}$ influence this crossover, rendering this check very difficult. To bypass this problem, it is tempting to imagine an alternative scenario, viz., the existence of the replica symmetric phase with nonzero order parameter even in zero field. This two-step process from the paramagnet to the RSB phase is present in mean field, but only for finite, albeit infinitesimal, n [17]. If this scenario occurred in low enough dimensions even for $n = 0$, the crossover from one type of transition to the other would disappear. In this case, the intermediate replica symmetric phase with $Q \neq 0$ would have the resemblance to a dropletlike phase. It is clearly necessary to perform further investigations, mainly a higher order calculation and numerical investigations.

In conclusion, we must stress that the theory we have put forward for the RSB transition is qualitatively different from the AT transition of mean-field theory. We argue that the change occurs at $d = 8$, below which the relevant Gaussian theory is that with zero replicon and anomalous masses, while infinite longitudinal one. It is this Gaussian fixed point which gives birth to the nontrivial one we found, governing the RSB transition below $d = 6$. We can speculate that these qualitative differences may affect the glassy phase too, resulting in a more general RSB scheme than the ultrametric one of mean-field theory.

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[23] You can find the fixed point (10) even if you start with $n = 0$; the RG flows, however, are ill-behaved around it: the Hessian is a nondiagonalizable matrix then, and the fixed point itself is not an isolated one, but sitting on a line of *equivalent* fixed points. As a matter of fact, the limit $n \to 0$ and the iteration of the RG are *not* interchangeable around that fixed point. Just these mathematical findings led us to throw away the condition of degeneracy between the L and A modes. From the physical point of view, the assumption of the divergence of the second term in the denominator of Eq. (6), for nonzero n, is the crucial deviation from conventional thinking about the AT transition. It was implicit in Ref. [13] that the replicon combination is the only correlation function divergent at criticality.